Predictive Density Combinations with Dynamic Learning for Large Data Sets in Economics and Finance

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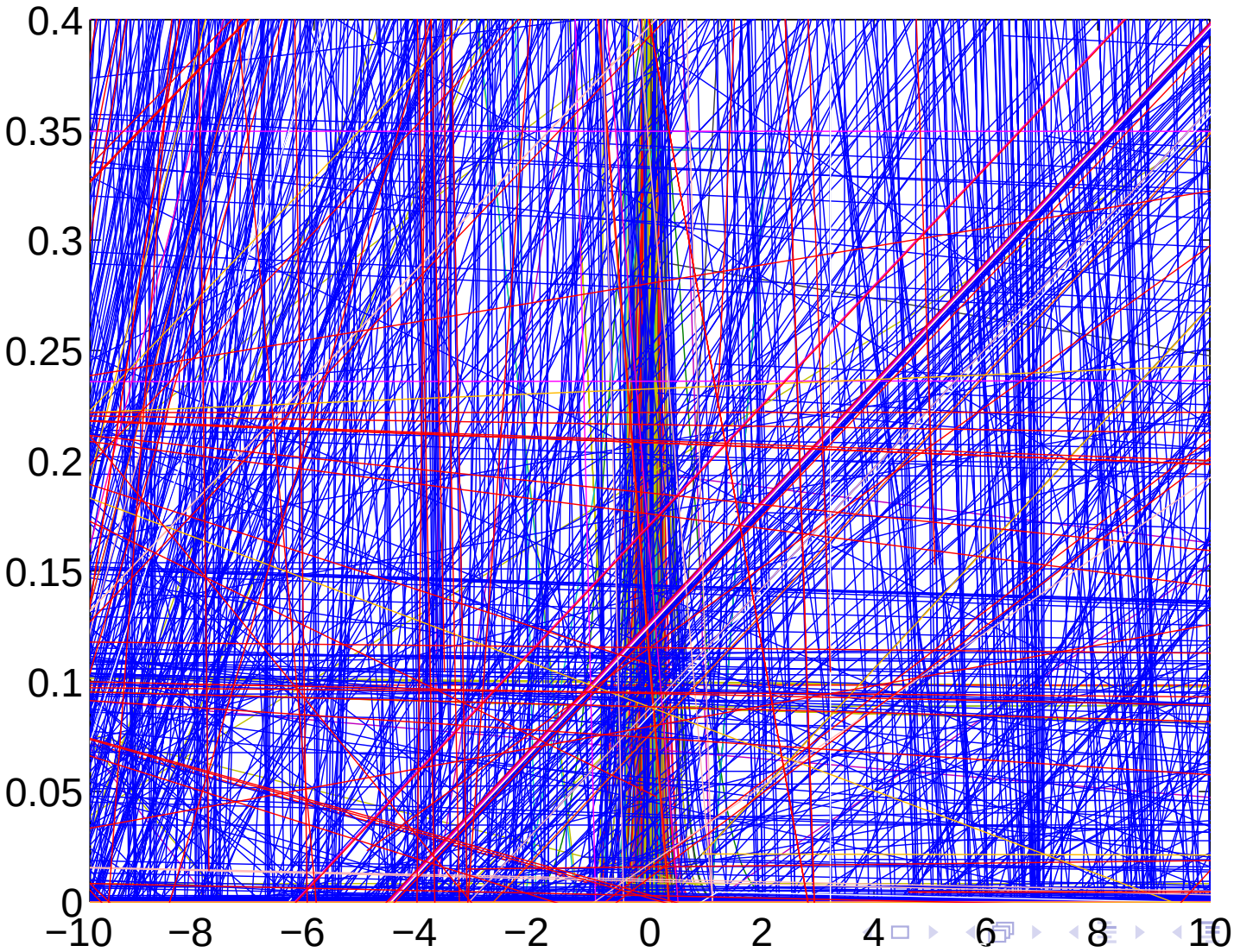
Motivation: Average 7424 Density Forecasts of 1856 Stock Returns



Motivation

- **Problem and Practice** Many agencies handle this averaging informally. Our aim: Give this a **Bayesian probabilistic foundation** in order to evaluate practical issues like: Probabilities of (extreme) events : Recession probability; Turning point probability; Probabilistic warnings about defaults; Value-at-Risk etc etc.
- **Fast growth in Big Data** give more accurate measures. Analogy with weather forecasting using many satellite pictures. But multimodality, skewness etc in economics.
- **Parallel Computing: New Hardware and Software** give openings to solve complex problems. **Machine learning with several hidden layers using neural networks have a direct connection with filtering methods in nonlinear time series models**.

Three Theoretical Contributions

- **Flexible Bayesian Combination Model is extension of Mixture of Experts** Model, (Jordan and Jacobs, 2010 and many others) allowing for cross-section and time dependent **Bayesian weight learning** and **diagnostic learning about model incompleteness**.
- **Dimension reduction in three steps**
  - **Sequential clustering** of large set of predictions with learning weights.
  - **Dynamic learning** of weights on a **simplex of reduced dimension**. (Time series behaviour on a bounded domain).
  - **From small simplex to large simplex** using **class-preserving property** of the logistic-normal distribution.
- **Model Representation and Efficient Computation**
  - Model is a **nonlinear State Space Model**
  - **K-means clustering algorithm** and **Nonlinear Sequential Filtering**. Filtering is parallelised and directly connected to hidden layers in machine learning with neural network weights

Review of the literature

- Averaging for forecast accuracy (Barnard (1963), Bates and Granger (1969)).
- Parameter and model uncertainties importance (BMA, Roberts (1965)).
- Probabilistic predictions provide larger set of information (Wallis).
- Correlations between forecasts and weights (Garratt, Mitchell and Vahey (2012)).
- Model performances differs over regions of interest/quantiles (mixture of predictives; generalized LOP: Fawcett, Kapetanios, Mitchell and Price, 2014; subsets of interest: Pelenis, 2014).
- Model performance varies over time possibly with persistence (Diebold and Pauly (1987), Guidolin and Timmermann (2009), Hoogerheide et al. (2010), Gneiting and Raftery (2007); Billio et al. (2013); Del Negro, Hasegawa and Schorfheide, 2015).
- Models might perform differently for multiple variables of interest (specific weight for each series, univariate models).
- Model set is possible incomplete (Geweke (2009), Geweke and Amisano (2010), Waggoner and Zha (2010)).
- GPU computing improve speed, see Casarin, et al. (2014), Dziubinski and Grassi(2013), Geweke and Durham (2012) and many others.

Review earlier density combination

- Billio, Casarin, Ravazzolo and Van Dijk (2013, JE) propose a **distributional state-space representation** of the predictive densities and the combination scheme
- Casarin, Grassi, Ravazzolo and Van Dijk, (2015, JSS) created **a MATLAB toolbox DECO** for estimating the density combination scheme of BCRVD (2013).
- Extensions to **Nowcasting** in Aastveit, Ravazzolo and Van Dijk (2018, JBES); **Forecasting Combinations and Portfolio Combinations** in Basturk, Borowska, Grassi, Hoogerheide and Van Dijk (2018). In this paper: **Large Data**
- **Bayesian Foundational** paper McAllinn and West (2018)
- Background: **Evolution of Density Combinations in Economics**, Aastveit, Mitchell, Ravazzolo and Van Dijk(2018)
- software on `http://www.francescoravazzolo.com/pages/DeCo.html`

Mixture representation extends mixture of experts model

Basic Idea for one economic variable y_t

- **Basic practice** $\sum_{i=1}^{n} w_{it}\tilde{y}_{it}$.
- **Formally: conditional predictive probability** of y_t, given $\tilde{\mathbf{y}}_t = (\tilde{y}_{1t}, \ldots, \tilde{y}_{nt})'$, is **discrete mixture of conditional predictive probabilities** of y_t given $\tilde{y}_{it}$, $i = 1, \ldots, n$ from n individual models with weights, w_{it}, $i = 1, \ldots, n$. **Fundamental density combination**

$$f(y_t|\tilde{\mathbf{y}}_t) = \sum_{i=1}^{n} w_{it} f(y_t|\tilde{y}_{it})$$

- **Discrete/continuous mixture representation**
  Under standard regularity conditions the **marginal predictive density of** y_t has the following discrete/continuous representation:

$$f(y_t|I) = \sum_{i=1}^{n} w_{it} \int_{\mathbb{R}} f(y_t|\tilde{y}_{it}) f(\tilde{y}_{it}|I_i) d\tilde{y}_{it}$$

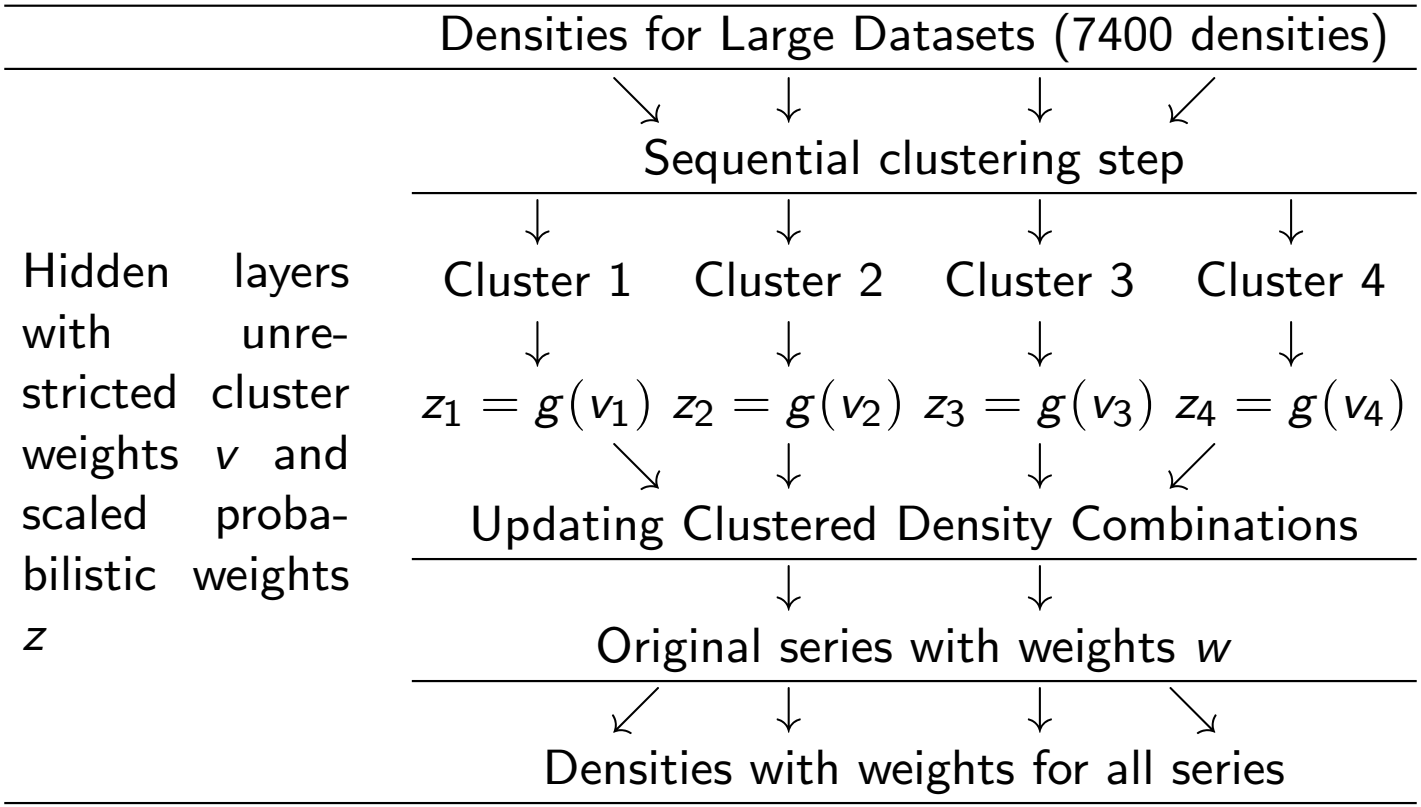
Mixture representation with learning about model incompleteness and dynamic weight behaviour

- We specify a Gaussian combination density $f(y_t|\tilde{y}_{it}) = \mathcal{N}\left(y_t|\tilde{y}_{it}, \sigma_t^2\right)$ with stochastic volatility, σ_t^2 **that determines overall uncertainty and indicates model incompleteness.** If σ_t^2 tends to zero, one obtains the static mixture of experts model from Jacobs and Jordan and Geweke and Keane. Also choice of the model set is important.
  This leads to **diagnostic learning.**
- Dynamic weights are **periodically updated with Bayesian learning**. (Here simple random walk).

Number of predictions is large, say n, reduce this to a small set, say m. Dimension reduction in three steps

- **Step 1: Sequential clustering of predictions to m clusters** with learning following some features of the predictive densities. Grouping can change over time. Weights driven by a model-specific predictive performance measure like log score (or equal weights).
- **Step2: Unrestricted weights v_t of clusters** follow a m-variate normal random walk process or more involved **Bayesian learning** and **are mapped to low dimensional simplex to have probabilistic interpretation of model weights, z_t**
- **Step 3: Map weights z_t from small $m-1$ simplex back to large $n-1$ simplex to w_t using the class-preserving property of the logistic-normal distribution** from **Aitchinson's geometry of the simplex** so that typical feature of model probability weight is preserved.

Summary of Data Driven Density Combinations for Large Datasets and connection with Machine Learning



Results

- **Hidden layer weights integrated using filtering. Also neural networks from machine learning.**
- Mixtures on **large space have same properties as reduced space.**

Representation result: (I) Nonlinear SSM as finite mixture with large sets of members and transition function as logistic-normal weight

Model: **marginal** predictive density of y_t with normal combination density, time-varying volatility for model incompleteness; dynamic learning of clustered weights

$$f(y_t|I) = \sum_{i=1}^{n} w_{it} \int_{\mathbb{R}} \mathcal{N}\left(y_t|\tilde{y}_{it}, \sigma_t^2\right) f(\tilde{y}_{it}|I_i) d\tilde{y}_{it}$$

Nonlinear State Space Model

$$\begin{aligned} \mathbf{y}_t &\sim \sum_{i=1}^{n} w_{it} \mathcal{N}\left(\tilde{y}_{it}, \sigma_t^2\right) && (1) \\ \tilde{\mathbf{w}}_t &\sim \mathcal{L}_{n-1}\left(\tilde{B}_t D_m \mathbf{v}_{t-1}, \tilde{B}_t D_m \Sigma D_m' \tilde{B}_t'\right), && (2) \end{aligned}$$

$\tilde{\mathbf{w}}_t = (w_{1,t}, \ldots, w_{n-1,t})'$ and $w_{n,t} = 1 - \tilde{\mathbf{w}}_t' \boldsymbol{\iota}_{n-1}$ and $\tilde{B}_t$ contains weights.

Representation result: (II) Under regular conditions the Model is Generalized Linear Model with Nonlinear Local Level Model

Corollary

Let $\mathbf{s}_t$ be an allocation vector, with $\mathbf{s}_t \sim \mathcal{M}_n(1, \mathbf{w}_t)$, where $\mathcal{M}_n(1, \mathbf{w}_t)$ denotes the multinomial distribution, and let σ_t be a time-varying variance. Then, the state space model given in the Proposition can be written as

$$\mathbf{y}_t = \tilde{\mathbf{y}}_t' \mathbf{s}_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(\mathbf{0}, \sigma_t) \tag{3}$$

$$\mathbf{s}_{i,t} = \begin{cases} 1 & \text{with probability } w_{i,t} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$\mathbf{w}_t = \boldsymbol{\phi}_{\boldsymbol{B}}(\mathbf{z}_t) \tag{5}$$

$$\mathbf{z}_t = \mathbf{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \mathcal{L}_{m-1}\left(\mathbf{0}, D_m \Sigma D_m'\right) \tag{6}$$

where $\boldsymbol{\phi}_{\boldsymbol{B}}(\mathbf{z}_t)$ is a nonlinear logistic transformation

Two Numerical approximations (prior is diffuse but proper)

- (1) Clustering-based mapping of the predictors requires the solution of an optimization problem which is not available in analytic form;
- (2) Analytic solution of the optimal filtering problem is generally not known;
- Apply sequential numerical approximation algorithms which, at time t iterate over the following steps:

1. **Parallel k-means clustering on GPU,** based on features $\boldsymbol{\psi}_{it}$ where the centroids are updated dynamically as follows:

$$\mathbf{c}_{jt+1} = \mathbf{c}_{jt} + \lambda_t(\mathbf{m}_{jt+1} - \mathbf{c}_{jt})$$

where

$$\mathbf{m}_{jt+1} = \frac{1}{n_{jt+1}} \sum_{i \in N_{jt+1}} \boldsymbol{\psi}_{it}$$

and $\lambda_t \in [0, 1]$. This implies a sequential clustering with forgetting driven by the processing of the blocks of observations.

2. **Sequential (over time dimension) Monte Carlo approximation** of the nonlinear SSM using GPU (more later)

Two Groups of Empirical Contributions

- **Empirical 1**: Financial Time Series: Data are 1856 individual stock prices, quoted in NYSE and NASDAQ. Predict Features of a Replication of S&P500 index.
  - **Better accuracy than standard models about many density features: means, volatilities and, in particular, tails.**
  - **Learning about** time behaviour of clusters of stocks. **Joint** dependence over time among weights. Signal of model incompleteness from diagnostic learning.
  - Prediction of the **statistical and economic accuracy of a tail estimates: event like Value-at-Risk**.

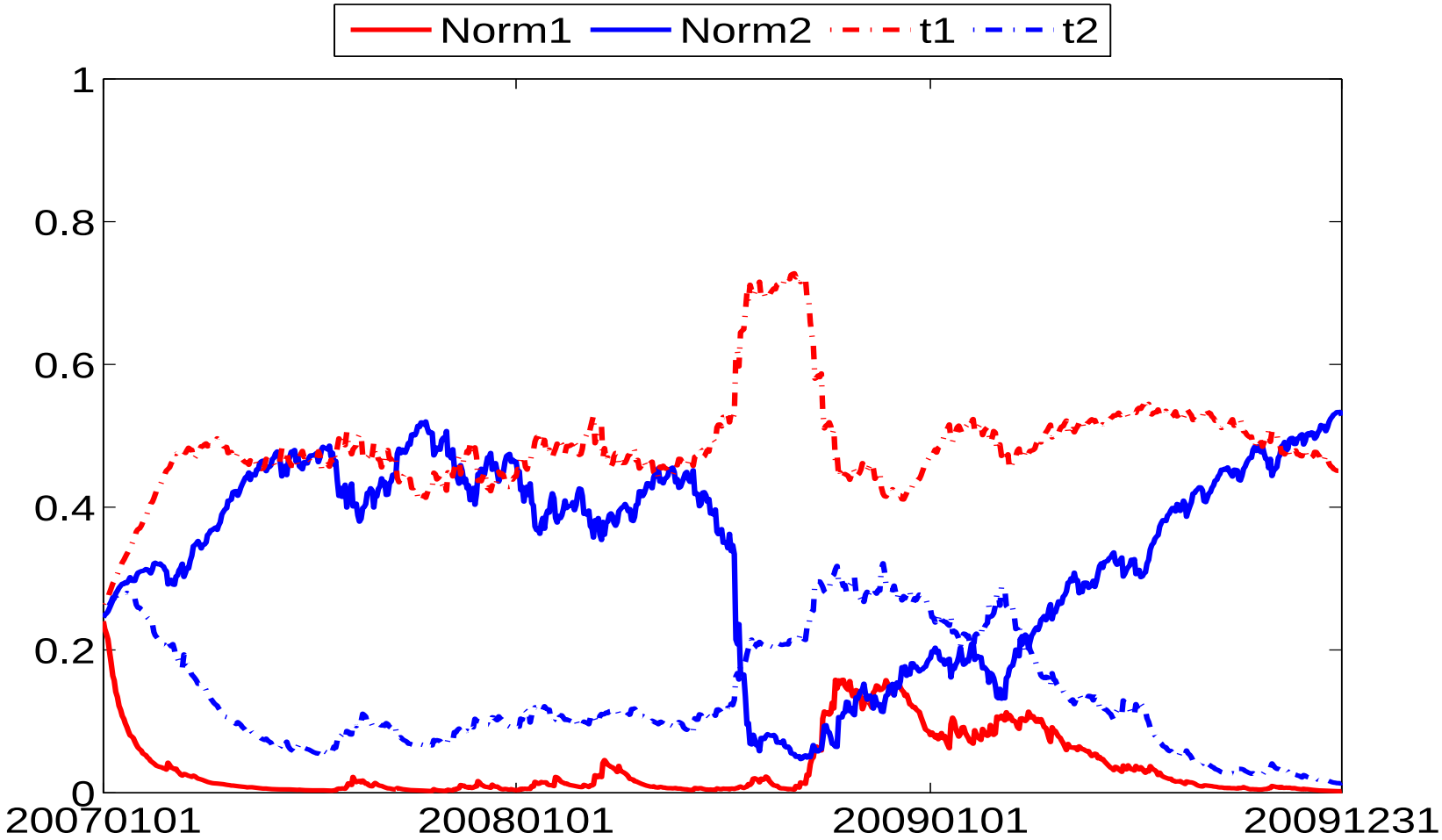
Empirical 1: Financial time series: Models and Estimation

- Data: 1856 individual stock daily prices quoted in the NYSE and NASDAQ from Datastream over the sample March, 18, 2002 to December, 31, 2009.
- Models and Estimation: Estimate a Normal GARCH(1,1) model and a t-GARCH(1,1) model via posterior mode using rolling samples of 1250 trading days (about five years) for each stock return:

$$\begin{aligned} y_{it} &= c_i + \kappa_{it}\varepsilon_{it} \\ \kappa_{it}^2 &= \theta_{i0} + \theta_{i1}\varepsilon_{i,t-1}^2 + \theta_2\kappa_{i,t-1}^2 \end{aligned}$$

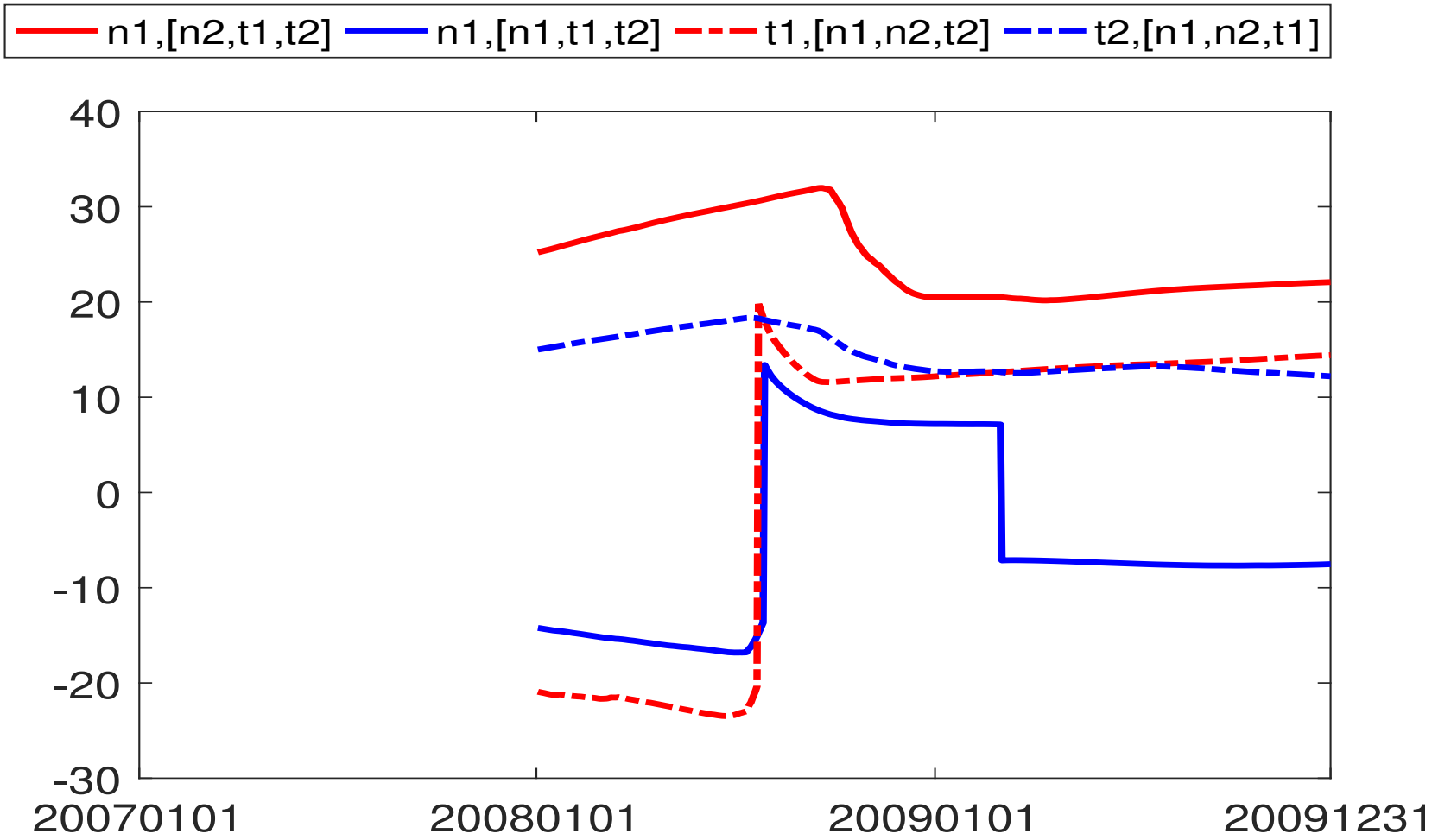
  where $y_{i,t}$ is the log return of stock i at day t, $\varepsilon_{it} \sim \mathcal{N}(0,1)$ and $\varepsilon_{it} \sim \mathcal{T}(\nu_i)$ for the Normal and Student-t cases, respectively.
- Produce 784 one day ahead density forecasts for the period January 1, 2007 to December 31, 2009.

Empirical 1: Dynamic behaviour of weights of 4 clusters



Normal GARCH(1,1) with **low (cluster 1) vol.** and **high (cluster 2) vol.**; t-GARCH(1,1) with **low (cluster 3)** and **high (cluster 4) d.o.f.**. Three subperiods: Weights adjust to time instability: Lehman Brothers Default

Empirical 1: Time movement of joint dependence of weights using Canonical Correlations



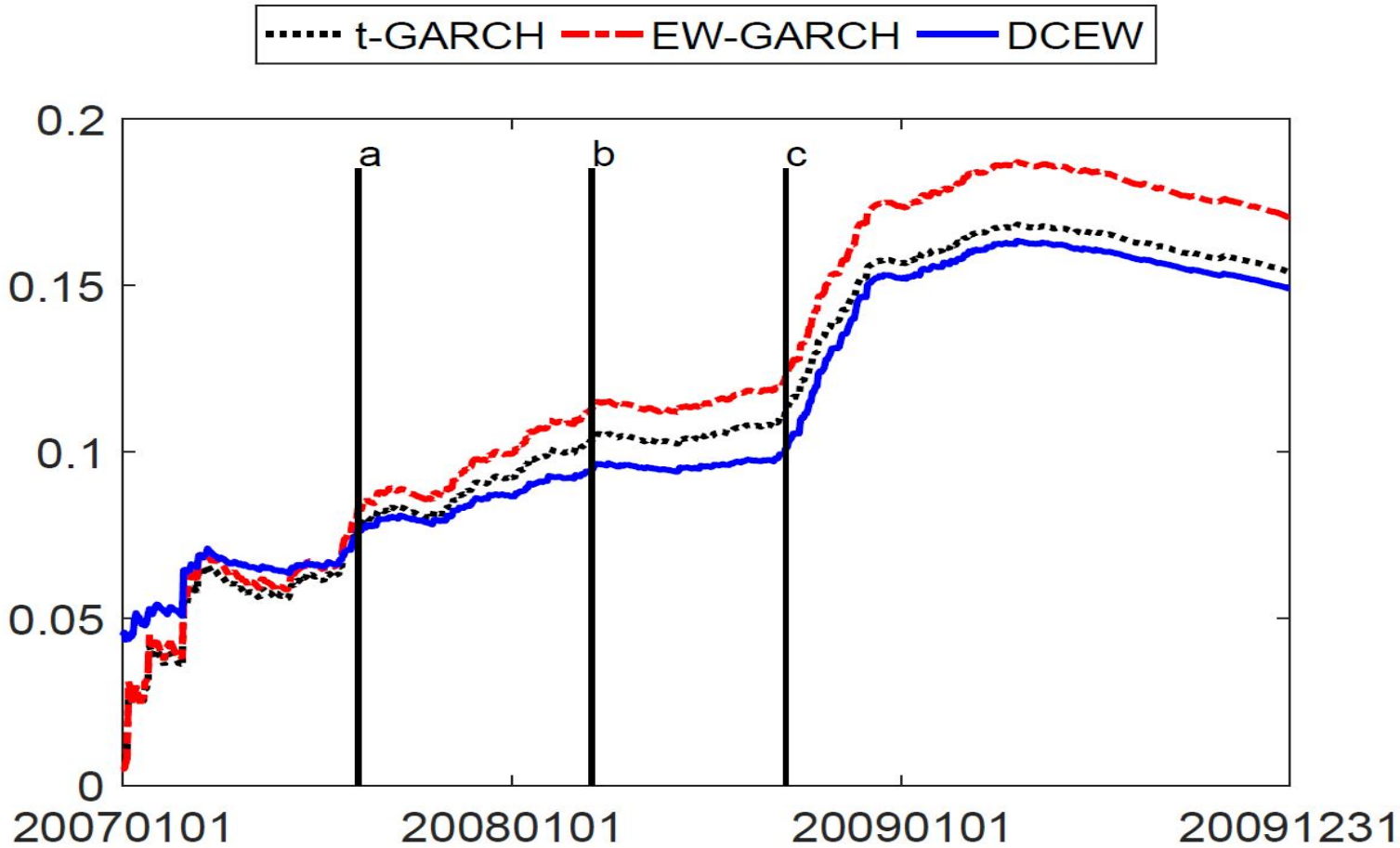
Each cluster versus all others. Change in cluster 2 and 3 after Lehman

Empirical 1: Forecasting Features of Densities: Means; Log Scores: Tail probabilities and Risk

	RMSPE	LS	CRPS	avQS-T	avQS-L	Violation
WN	1.852	-9.045	1.017	0.429	0.425	3.57%
Normal GARCH	1.852	-4.164**	0.956**	0.139**	0.195**	2.93%
t-GARCH	1.852	-2.738**	0.937**	0.118**	0.154**	2.55%
GJR-GARCH	1.852	-4.068**	0.955**	0.125**	0.158**	2.75%
EW-GARCH	1.853	-3.145**	1.018	0.144**	0.171**	2.80
DCEW	**1.812****	**2.249****	**0.911****	**0.114****	**0.149****	0.90%
DCEW-SV	1.816**	2.206**	**0.913****	0.114**	0.149**	**1.02%**

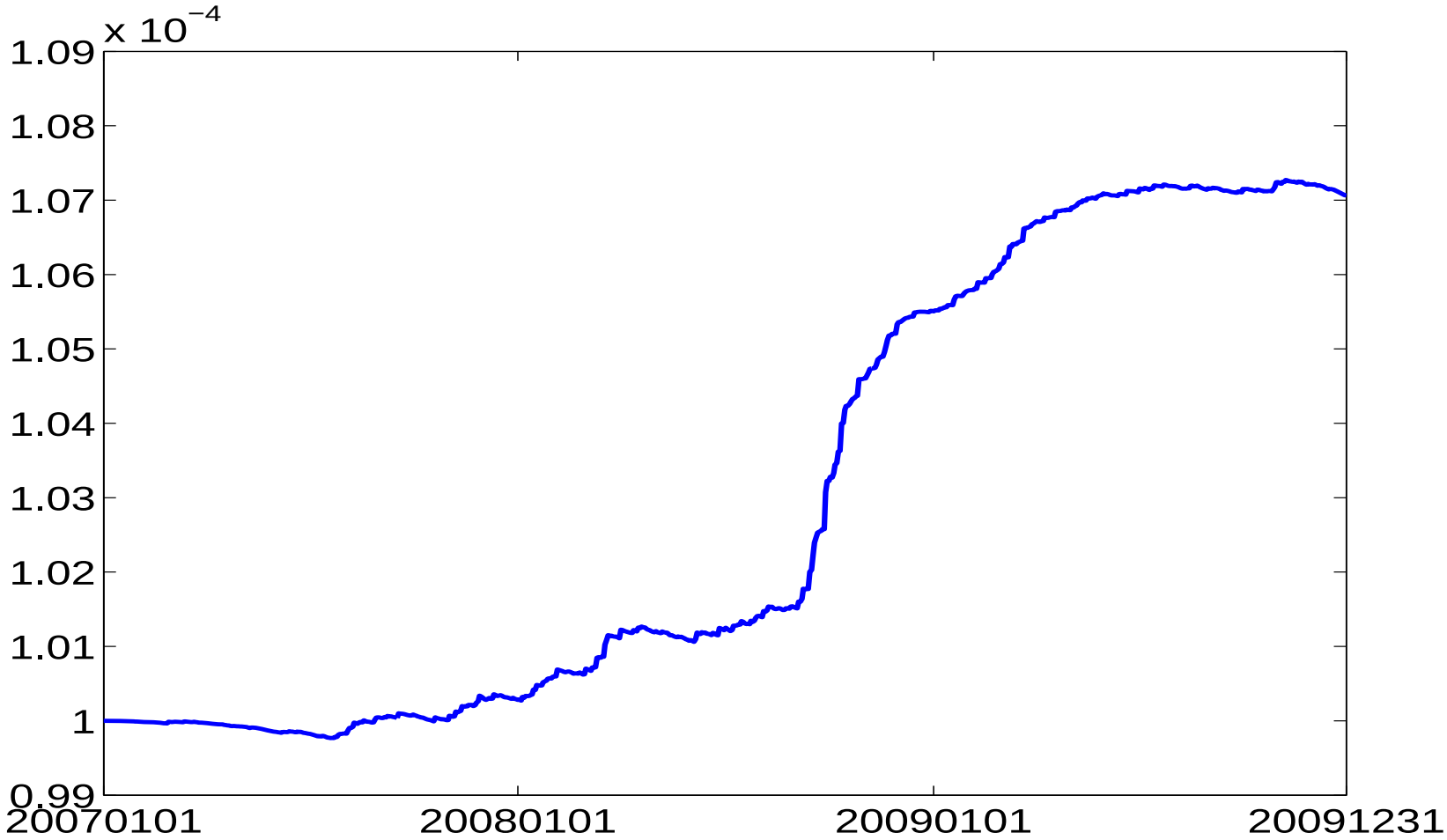
Table: Forecasting results for next day S&P500 log returns.
avQS-T and avQS-L: average tails (T) and average left tail (L) scores described in Gneiting and Raftery (2007).
Violation: percentage of times the realization exceeds the 1% Value at Risk (VaR).

Empirical 1: Cumulative left quantile score for 3 cases



Timeline legend: a - 8/9/2007, BNP Paribas redemptions on three investment funds; b - 3/17/2008, collapse of Bear Stearns; c - 9/15/2008, Lehman bankruptcy. Stock market stress increases gap between models. Accuracy drops after Lehman default

Empirical 1: Model incompleteness measured by σ_t



Variance increases in DCEW-SV scheme in September 8, 2008. Signal for including possibly models with jumps in volatility

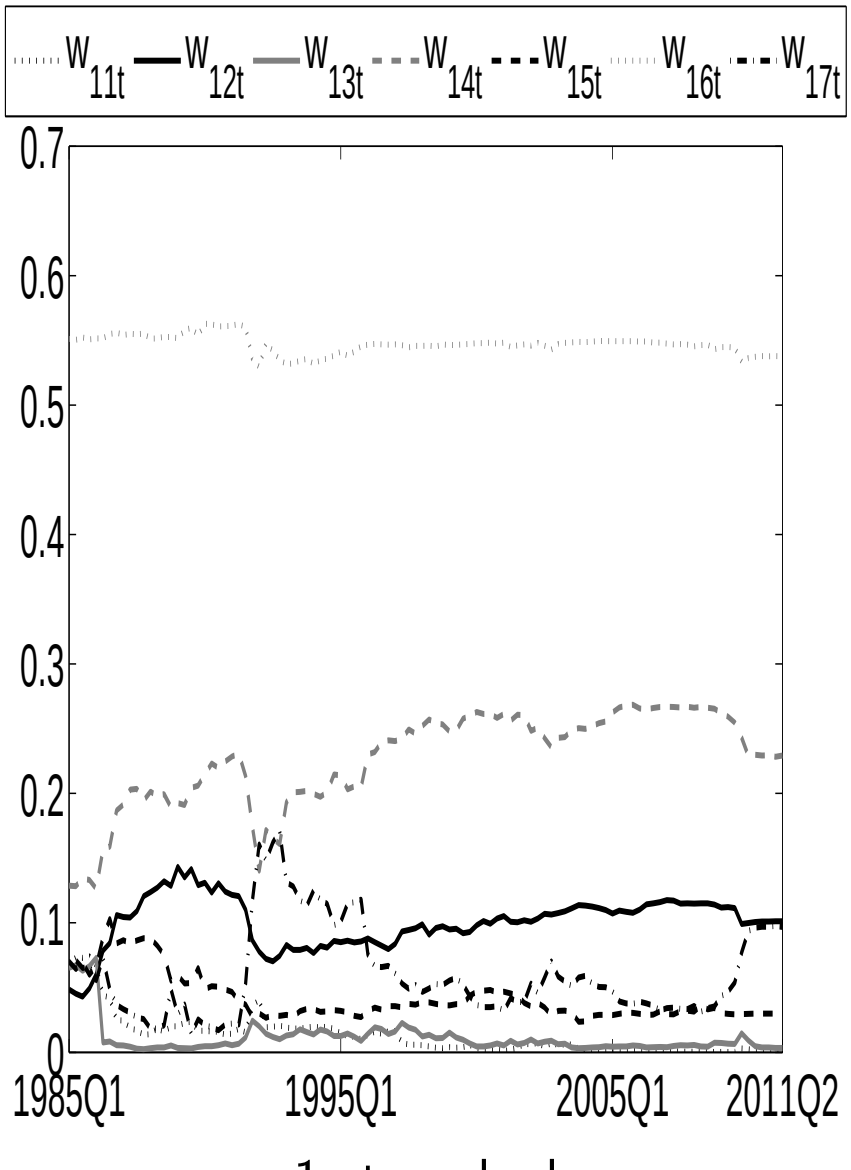
Empirical 2: Stock and Watson (2005) dataset. Models and Estimation GDP, GDP deflator, 3-month Interest rates, Employment

- **Empirical 2**: Macroeconomic Time Series. Extend Stock and Watson (2002,2005) as follows:
  - **Joint prediction model** for the group of variables Real GDP, GDP deflator, Treasury Bill rate, Employment instead of a single variable 'beats' all alternatives considered.
  - **Learning about dynamic behaviour of 5-7 sectors**. Factor-model combination shows rise and decline of sectors. Dominant factor?
  - **Accurate probability of turning point and of recession measured over time**

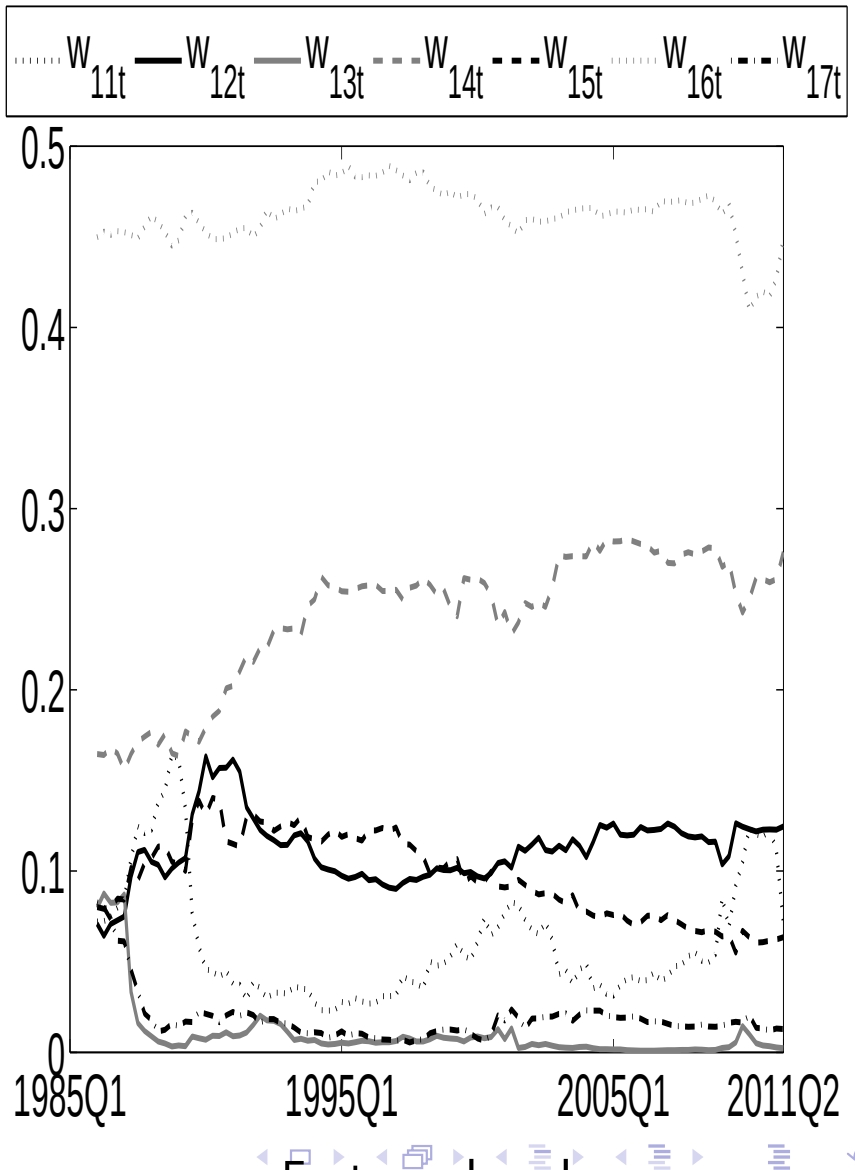
Empirical 2: Stock and Watson (2005) dataset. Models and Estimation GDP, GDP deflator, 3-month Interest rates, Employment.

- Stock and Watson (2005) dataset: 142 series, standardized and sampled at a quarterly frequency from 1959Q1 to 2011Q2.
- Bayesian estimation of AR(1) process for all series and group them using residual variance and the persistence parameters.Identify $m = 5, 7$ clusters.
- Combination: univariate and multivariate combinations, 5 and 7 clusters, equal and log score weights.
- Produce from 1 to 5-step ahead recursive AR(1) forecasts for the out-of-sample: 1985Q1-2011Q2.
- Evaluation criteria: MSPE, CRPS and Log Score.

Empirical 2: Dynamic behaviour of cluster weights of GDP growth: dominant is 6 (Exports, Imports, GDP deflator)

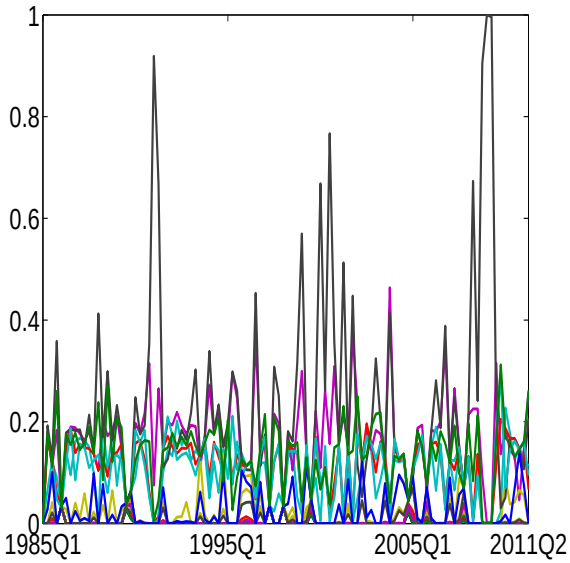


1-step ahed

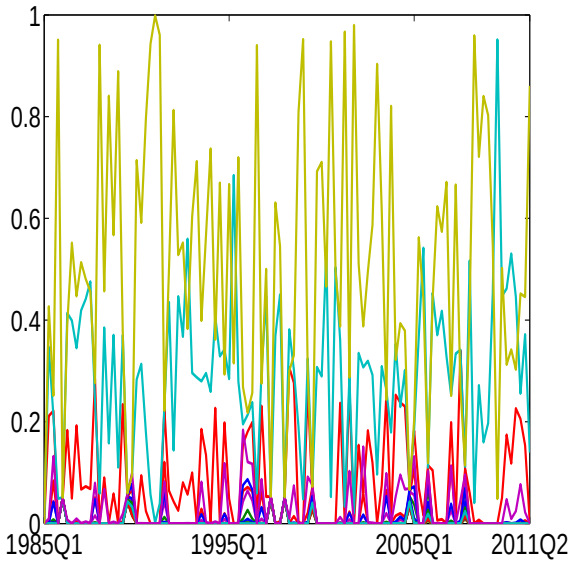


5-step ahead

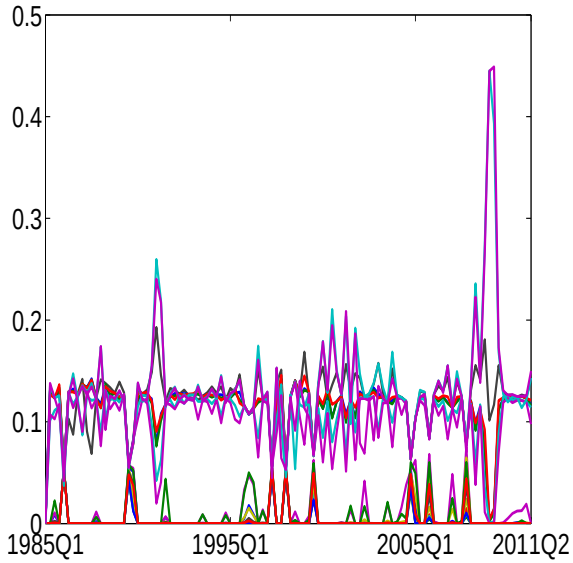
Empirical 2: Dynamic behaviour of model weights w of GDP growth, 1-step ahead



cluster 1



cluster 2



cluster 3

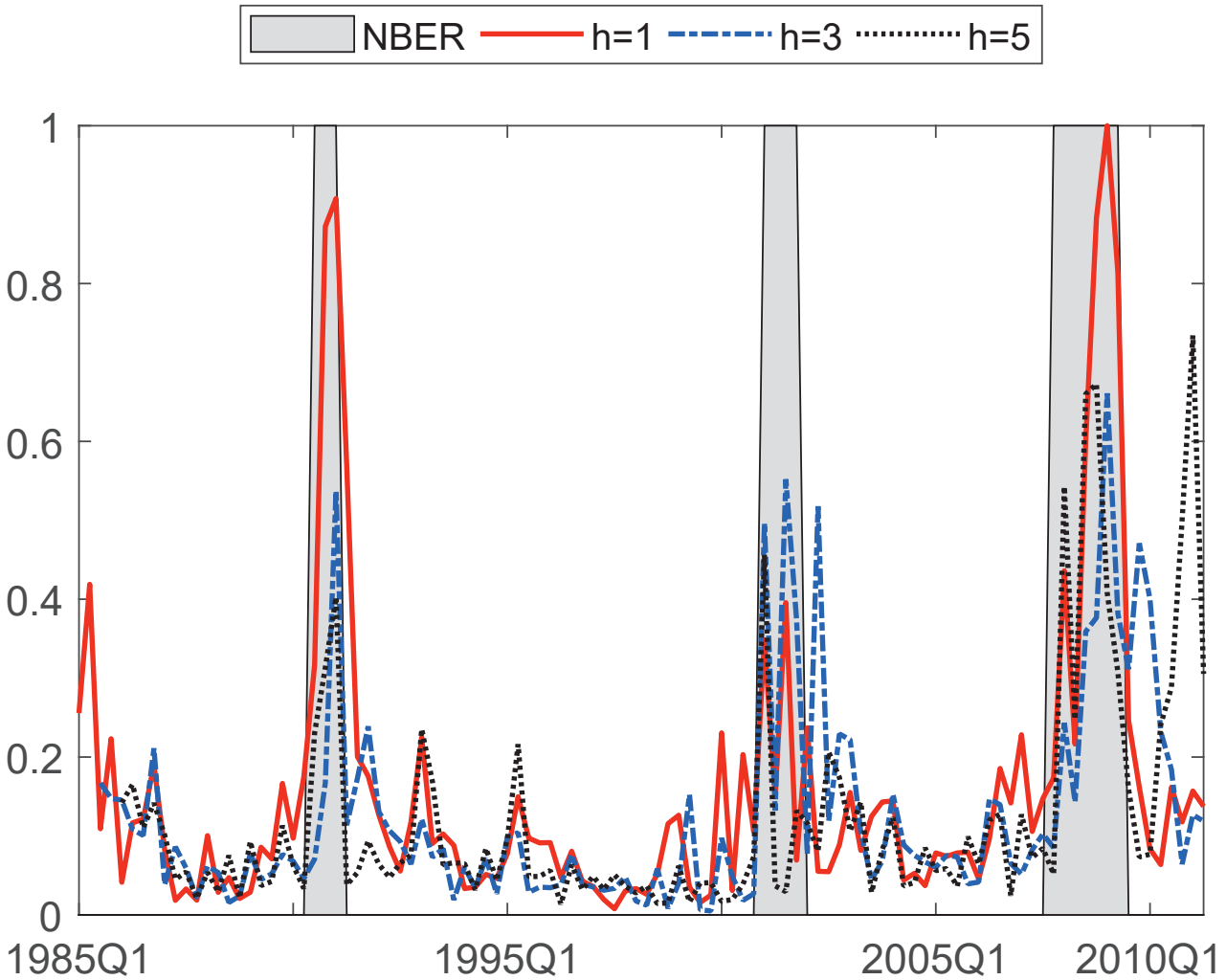
Empirical 2: 8 Competitive models for Forecasting

- UDC versus MDC (univariate versus multivariate combination)
- EW versus LS (equal weights versus recursive log score weights);
- 5 versus 7 (5 cluster versus 7 clusters);
- 8 models: UDCEW5, MDCEW5, UDCLS5, UDCLS7, MDCLS5, UDCEW7, MDCEW7, UDCLS7, MDCLS7

Empirical 2: GDP Forecasting Results

	h=1			h=5		
	MSPE	LS	CRPS	MSPE	LS	CRPS
AR	0.647	-1.002	0.492	0.682	-1.009	0.506
BDFM	0.649	-1.091	0.382**	0.655	-1.099	0.388**
UDCEW5	0.644	-0.869	0.333**	0.658*	-0.912	0.343**
MDCEW5	0.63	-0.928	0.326**	0.636*	-0.844	0.324**
UDCLS5	0.773	-1.306	0.464	0.715	-1.38	0.481
MDCLS5	0.725	-1.145	0.505	0.557*	-1.005	0.358**
UDCEW7	0.649	-0.875	0.334**	0.657*	-0.891	0.338**
MDCEW7	0.642	-0.979	0.334**	0.654*	-1.009	0.342**
UDCLS7	0.646	-0.868*	0.332**	0.657*	-0.914	0.342**
MDCLS7	**0.596***	**-0.586****	**0.275****	**0.610****	**-0.634****	**0.286****

Empirical 2: Probabilities of negative quarterly growth



One quarter ahead, three quarters ahead and five quarter ahead probabilities over time of negative quarterly growth given by the the combination approach and ex-post NBER

Conclusions

- **Time-varying combinations** of large sets of predictive densities based on **sequential clustering** analysis can deal with **big data**.
- Combination weights are driven by **cluster-specific latent processes much smaller** than the number of available predictors.
- **The proposed model is a nonlinear SSM with finite mixtures and dynamic logistic-normal weights. Density combination evaluated by nonlinear sequential filtering. Interesting connections with machine learning with neural nets**
- Forecasting financial time series: Three improvements: **Forecast accuracy of moments; dynamic learning about clusters; efficient estimates of risk**
- Substantial gains in point and density forecasting of **joint** set of US real GDP, GDP deflator, Treasury Bill returns and employment growth based on log score learning. **Rise and decline of sectors; probabilities of recession.**
- Parallelisation:Handling big data sets with GPU can be easy and fast.

Topics for Further Research

- **Modelling and estimation**. Use **more information** on clustering and explore diagnostic analysis about model incompleteness and richer model set.
- More on **efficiency of filter** methods: New Filter obtained using the mixture of student density approximation constructed using an EM weighted importance sampling approach, HOVD(2012, JE), labeled M-Filter. Forthcoming Journal of Econometrics.
- **Forecasting and Policy**. More on Nowcasting and Multiperiod out-of-sample forecasting. Applications to **Policy Issues**: In the field of Finance using Decision Models. Current research on Bayesian Dynamic Modelling and Time-varying combinations of Equity Momentum Strategies using US industrial portfolios 1929-2015. **Time-varying combinations jointly for models and policies: mixture of mixtures**. Challenge to do this for macro-models.
- **More efficient parallel computing.**