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Alejandro Van der Ghote

Benefits of macro-prudential policy in low interest rate environments

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Abstract

I study macro-prudential policy intervention in economies with secularly low interest rates. Intervention boosts risk-free real interest rates unintentionally, simply as a by-product of containing systemic risk in financial markets. Thus, intervention also boosts the natural rate of return in particular (i.e., the equilibrium risk-free rate that is consistent with inflation on target and production at full capacity). These results point to a novel complementarity between financial stability and macroeconomic stabilization. Complementary is sufficiently strong to generate a divine coincidence if the natural rate is secularly low, but not too low.

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Keywords: Macro-prudential Policy, Systemic Risk, Natural Rate of Return

Non-technical Summary

In economies with sufficiently low interest rates, macro-prudential policy has additional benefits apart from safeguarding financial stability. Notably, macro-prudential interventions that curb leverage during upturns also help manage aggregate demand during downturns. Farhi and Werning (2016) and Korinek and Simsek (2016) are the first to formalize these benefits. According to their argument, aggregate deleveraging depresses real interest rates and, if initially the rates were sufficiently low, it may also render the monetary policy rate constrained by the effective lower bound (ELB) on nominal interest rates. In addition to financial distress from aggregate deleveraging, an artificially high policy rate, a subdued aggregate demand, and low inflation below target may then ensue. Put shortly, the economy may also enter into a liquidity trap. In such environments, the argument concludes, add-on policy interventions that further restrict leverage during upturns also helps reduce the possibility of the trap or its severity should the trap occur.

This paper reveals another mechanism through which macro-prudential policy helps sustain macroeconomic stabilization in low interest rate environments. The mechanism instead operates through risk premia. By containing systemic risk in financial markets, macro-prudential policy also contains systematic risk in the economy. As a consequence of the systematic risk reduction, the insurance value of risk-free claims falls, which boosts risk-free real interest rates in general and the natural rate of return in particular (i.e., the equilibrium risk-free rate that is consistent with inflation on target and production at full capacity). A higher natural rate, it follows, helps mitigate the intensity of liquidity traps, especially during turbulent financial times when systemic risk is unusually high.

The paper also reveals the possibility of a divine coincidence between financial stability and macroeconomic stabilization. The divine coincidence exists only if the following two conditions are met. First, the natural rate under a macro-prudential policy of laissez faire falls below the ELB at least occasionally over the cycle. Second, the same rate but under a macro-prudential policy that is concerned only with financial stability instead lies above throughout. If the two conditions hold, macro-prudential policy is also essential for sustaining macroeconomic stabilization, and it helps achieve this other policy objective even unintentionally, simply as a by-product of safeguarding the stability of the financial system.

1 Introduction

In economies with sufficiently low interest rates, macro-prudential policy has additional benefits apart from safeguarding financial stability. Notably, macro-prudential interventions that curb leverage during upturns also help manage aggregate demand during downturns. Farhi and Werning (2016) and Korinek and Simsek (2016) are the first to formalize these benefits. According to their argument, aggregate deleveraging depresses real interest rates and, if initially the rates were sufficiently low, it may also render the monetary policy rate constrained by the effective lower bound (ELB) on nominal interest rates. In addition to financial distress from aggregate deleveraging, an artificially high policy rate, a subdued aggregate demand, and low inflation below target may then ensue. Put shortly, the economy may also enter into a liquidity trap. In such environments, the argument concludes, add-on policy interventions that further restrict leverage during upturns also helps reduce the possibility of the trap or its severity should the trap occur.

In this paper, I reveal another mechanism through which macro-prudential policy helps sustain macroeconomic stabilization in low interest rate environments. The mechanism instead operates through risk premia. By containing systemic risk in financial markets, macro-prudential policy also contains systematic risk in the economy. As a consequence of the systematic risk reduction, the insurance value of risk-free claims falls, which boosts risk-free real interest rates in general and the natural rate of return in particular (i.e., the equilibrium risk-free rate that is consistent with inflation on target and production at full capacity). A higher natural rate, it follows, helps mitigate the intensity of liquidity traps, especially during turbulent financial times when systemic risk is unusually high.

I also reveal the possibility of a divine coincidence between financial stability and macroeconomic stabilization. The divine coincidence exists only if the following two conditions are met. First, the natural rate under a macro-prudential policy of laissez faire falls below the ELB at least occasionally over the cycle. Second, the same rate but under a macro-prudential policy that is concerned only with financial stability instead lies above throughout. If the two conditions hold, macro-prudential policy is also essential for sustaining macroeconomic stabilization, and it helps achieve this other policy objective even unintentionally, simply as a by-product of safeguarding financial stability. The coincidence is typical of environments with secularly depressed, but not too depressed, natural rates. If the second condition does not hold, complementarity between the policy objectives still exists, but it is not sufficiently strong to generate the coincidence. In this other scenario, macro-prudential policy further compresses systemic risk relative to what it would do in the counterpart economy without the ELB—even if its single objective is maximizing financial stability. The tighter policy further lifts the natural rate. Thus, it also further mitigates financial disruptions from artificially high funding rates and production capacity underutilization.

To formalize these ideas and findings, I use a general equilibrium model of financial intermediation with endogenous systemic risk in financial markets and endogenous risk in aggregate consumption. The model builds on the work of Brunnermeier and Sannikov (2014); Caballero and Simsek (2020); and Van der Ghote (2020). In the model, the aggregate net worth of financial intermediaries as a share of total wealth suffices to summarize the phase of the cycle. Systemic risk is inversely U-shaped in the wealth share and it peaks precisely when the wealth share is at intermediate values. In the intermediate phases, financial intermediaries as a whole are sufficiently well capitalized to have large aggregate effects, but not to tolerate adverse disturbances to their net worth without selling assets (i.e., securitized portfolios of loans to nonfinancial firms) or curtailing financing to the firms. The sale of assets dislocates asset prices; sparks a two-way feedback loop between falls in the prices and price-fueled losses of intermediary net worth, and, ultimately, exacerbates instability in the wealth share of the intermediaries, aggregate financing to nonfinancial firms, aggregate production capacity, and aggregate output. These events jointly happen because the agents who buy the liquidated assets (e.g., households) are less adept at managing portfolios of loans and financing firms.

In a first version of the model, I abstract away from liquidity traps and monetary and macro-prudential policy intervention. To do so, I consider a real economy under laissez faire. I proceed in this way to first examine the equilibrium relationship between systemic risk and the natural rate.

In this economy, spikes in systemic risk generate spikes in aggregate output risk and the risk premium for aggregate consumption risk. The natural rate thus tanks when systemic risk peaks. Systemic risk is indeed a key determinant of the lower bound on the rate. The bound can be negative even if the rate is positive in the frictionless version of the economy. At the edge of financial distress (i.e., when systemic risk is about to peak) the natural rate is rather stable. However, once the economy enters distress, fluctuations in the rate spike abruptly (i.e., discontinuously) and the rate remains highly unstable until the economy exits. Formally, the rate as a function of the state has a discontinuity at the edge point, which makes both the level and the variability of the rate jump at that point.

In a second version of model, I incorporate the traps and monetary policy, but still abstract away from macro-prudential intervention. I consider a monetary economy with fully rigid nominal prices and a zero lower bound (ZLB) constraint on nominal rates. The policy rate tracks the natural rate whenever possible and remains stuck at the ZLB otherwise. This rule can be regarded as a passive monetary policy that is concerned only with macroeconomic stabilization in the short term. I focus on the implications of the passive rule and the ZLB on the relationship between systemic risk and the natural rate.

In this other economy, liquidity traps occur when systemic risk peaks, and the natural rate falls into negative terrain. During the traps, the policy rate remains artificially high at zero, however, and aggregate demand is subdued. Economic activity is depressed, not only because parallel disruptions in financial markets destroy production capacity, but also because aggregate demand falls drastically (i.e., discontinuously) below the remanent capacity after destruction.

Following adverse disturbances to intermediary net worth, underutilization of (already impaired) production capacity, combined with artificially high funding rates, further compress intermediation margins beyond what is consistent with asset price dislocation alone. The additional setback worsens intermediary profitability and exacerbates instability in financial markets. It also increases systemic risk, which further depresses the natural rate and aggravates the liquidity trap. Relative to the counterpart economy without the ZLB constraint, systemic risk reaches even higher peaks, the natural rate reaches even lower troughs, and the interactions between systemic risk and the natural rate are stronger.

In the third, and last, version of the model, I also incorporate macro-prudential policy. I restrict attention to a state-contingent limit on leverage. This is the most standard macro-prudential instrument in actual economies. I consider the same passive monetary policy as in the second version. In the model, the policy-based limit is binding only when it is below both a market-based limit on and the "efficient" quantity of leverage (i.e., the quantity at which the intermediaries alone finance all of the firms). Macro-prudential policy can improve financial stability, macroeconomic stabilization, and social welfare over the financially unregulated economy. This is because of pecuniary externalities in financial intermediation and an aggregate demand externality. The complementarity and divine coincidence results above can indeed be restated in terms of the externalities. The divine coincidence exists if (*i*) the above first condition holds and (*ii*) the optimal regulation of the pecuniary externalities under the postulate of a nonexistent aggregate demand externality is consistent with the postulate in equilibrium. If the second condition does not hold, even a macro-prudential policy that is concerned only with financial stability intentionally curbs the aggregate demand externality. Regardless of whether the divine coincidence exists, moderating the pecuniary externalities alleviates the aggregate demand externality, which reveals a complementarity between the two.

Related Literature This paper is related primarily to two strands of the literature. First, it relates to a literature on the secular decline of natural rates in industrial economies. Two strong consensuses are that natural rates have been in a secular decline over the past three decades or so (Bernanke et al. (2005); and Caballero, Farhi and Gourinchas (2008); among others) and that secular declines have accelerated in the aftermath of the financial crisis of 2008-09 (Summers (2014a); Baldwin and Teulings (2014); and Rachel and Summers (2019), among others). A new consensus is burgeoning that natural rates will continue and further accelerate their secular decline going forward (Blanchard (2020); Jordà, Singh and Taylor (2020)). I take past, current, and expected future secular declines as a given, and focus instead on their consequences for the relationship between systemic risk and the natural rate. The model predicts an inverse relationship and, in particular, that the natural rate tanks when systemic risk peaks. These predictions are in line with empirical findings by Del Negro et al. (2019) and Kuvshinov and Zimmermann (2020). The model also generates interactions between liquidity traps, depressed growth trends, and instability in financial markets, which are consistent with Summers (2014b).

Second, this paper relates to a literature on the transmission channel of prudential policy in low interest environments. Notable papers in this literature include Farhi and Werning (2016); Korinek and Simsek (2016); and Fornaro and Romei (2019). In an open economy setup with many countries, Fornaro and Romei (2019) find that domestic interventions in line with those in Farhi and Werning (2016) and Korinek and Simsek (2016) may backfire internationally if all of the countries implement the interventions simultaneously. However, this result does not generally apply to the prudential transmission channel in this paper. This is because reductions in systemic risk within a country, in general, help reduce global financial risk, which in turn helps contain systemic risk in the other countries as well. Three other notable papers are Ferrero, Harrison and Nelson (2018); Caballero and Simsek (2019); and Rubio and Yao (2020). In those papers, boom-bust cycles evolve in a one-way direction from booms to busts, whereas in this paper, the cycles oscillate continuously throughout. The specification in this paper is thus better suited for environments in which liquidity traps are recurrent. Those environments are expected to remain going forward according to forecasts by Kiley and Roberts (2017) and Jordà, Singh and Taylor (2020).

Outline Section 2 lays out the baseline model. Section 3 solves that model and Section 4 discusses the main results. Section 5 incorporates monetary considerations and macro-prudential policy in the baseline model and conducts the policy analysis. Section 6 concludes.

2 The Baseline Model

The baseline model builds on work by Gertler and Kiyotaki (2010); Gertler and Karadi (2011); Brunnermeier and Sannikov (2014); Maggiori (2017); and Van der Ghote (2020). The focus in this model is on the relationship between systemic risk and the natural rate.

Agents A continuum of identical households and financial intermediaries populate the economy. These two types of agents differ inherently in that financial intermediaries have an edge over households in managing the single available real asset. The edge could be rationalized as resulting from a skill advantage of financial intermediaries at monitoring the activities of some nonfinancial firms that use the asset to produce (Van der Ghote (2020)). I refer to the asset as physical capital in what follows.

Technology Physical capital k_t yields output flows y_t per unit of time in terms of a final consumption good according to

$$y_t = a_t k_t \ , \tag{1}$$

with productivity a_t being either high $a_t = 1$ or low $a_t = a_h < 1$ depending on whether financial intermediaries or households manage the units of physical capital. High productivity equals 1 just as a normalization. Physical capital is the single risky asset. It evolves over time stochastically according to

$$\frac{dk_t}{k_t} = [I(\iota_t) - \delta]\, dt + \sigma dZ_t \ , \tag{2}$$

with dZ_t being a standard Brownian disturbance that satisfies the usual conditions, $\sigma > 0$ a positive parameter, and $[I(\iota_t) - \delta]\, k_t$ a standard investment net return function that satisfies $I' > 0$ and $I'' < 0$. Shock dZ_t is common across all of the units of physical capital and it can therefore be interpreted as an aggregate disturbance to the growth rate of the productive quality of physical capital (Brunnermeier and Sannikov (2014)). The investment net return function is the same for both financial intermediaries and households; it is such that investment rate ι_t has positive but decreasing marginal returns, and is net of depreciation costs δk_t, with $\delta > 0$ being a parameter. In Section 4, I will show that investment is not essential for interactions between systemic risk and the natural rate. However, I will also show that investment brings additional channels through which financial conditions and systemic risk interact with the rate. Investment expenditure flows total $\iota_t k_t$ per unit of time in terms of consumption. Output net of internal reinvestment expenditures thus is $y_t - \iota_t k_t = (a_t - \iota_t)\, k_t$. Time t is continuous.

Return on Physical Capital Physical capital is traded continuously in fully liquid markets at a spot price $q_t > 0$. The total rate of return on physical capital net of investment expenditures, therefore, is

$$dR_{e,t} \equiv \frac{[\mathbf{1}_{e=f} + (1 - \mathbf{1}_{e=f})\, a_h] - \iota_{e,t}}{q_t} dt + \frac{d(q_t k_t)}{q_t k_t}, \text{ with } e \in \{f, h\} \ , \tag{3}$$

with the first term on the RHS being the net dividend yield rate, the second term the capital gain/loss rate, and the difference between $dR_{f,t}$ and $dR_{h,t} < dR_{f,t}$ resulting from the managing advantage of financial intermediaries (f) over households (h). I postulate that in equilibrium, spot price q_t evolves over time stochastically, according to

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q,t} dZ_t, \text{ with } \sigma_{q,t} \geq 0 \ , \tag{4}$$

with dZ_t being the same Brownian disturbance that dictates the evolution of physical capital, and $\mu_{q,t}$ and $\sigma_{q,t}$ endogenous drift and diffusion processes, respectively, to be determined later. The postulate ensures that $dR_{f,t}$ and $dR_{h,t}$ respond on impact to shock dZ_t according to

$$dR_{e,t} = \left[\frac{[\mathbf{1}_{e=f} + (1 - \mathbf{1}_{e=f})\, a_h] - \iota_{e,t}}{q_t} + \mu_{q,t} + I(\iota_{e,t}) - \delta + \sigma_{q,t}\sigma \right] dt + (\sigma_{q,t} + \sigma)\, dZ_t \ . \tag{5}$$

This expression is indeed consistent with physical capital's being risky.[1]

[1]The expression follows from Ito's product rule. Formally, physical capital is locally risky during time interval $(t, t + dt)$, because its rate of return depends on the Brownian disturbance.

Deposits Besides physical capital, there are also deposits. Deposits are financial securities in zero-net supply; they are short-term, meaning that they mature at time $t + dt$, and they are allegedly noncontingent, meaning that they promise to repay during $(t, t + dt)$ a fixed rate of return $r_t dt$ that remains the same regardless of shock dZ_t. Deposits allow taking leveraged positions on physical capital. However, because of their difference in riskiness from physical capital, deposits bundle leverage with risk-taking. This bundling is one of the necessary elements in the setup for the existence of systemic risk. The other necessary element is a market-based portfolio constraint, which I will introduce next. Based on the return difference, $dR_{f,t} > dR_{h,t}$, in what follows, I postulate that in equilibrium financial intermediaries alone issue deposits and take leveraged positions on physical capital.

Market-based Portfolio Constraint The portfolio constraint follows from a limited enforcement problem in deposits markets that is similar to the problem in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). Specifically, immediately after issuing deposits, financial intermediaries can divert a fraction $1/\lambda \in (0, 1)$ of their assets. However, if the market value of the remanent assets after diversion is below their principal obligations, the intermediaries are legally bound to shut down. In equilibrium, the intermediary business has a value $V_t \geq n_{f,t} \geq 0$, with $n_{f,t} \geq 0$ being the net worth of the intermediary agent. Following Maggiori (2017), to ensure that this problem is also relevant in a continuous-time framework, I assume that financial intermediaries are each owned by a single household and that they can only issue deposits to households other than their owner. The problem thus shapes an incentive-compatible (IC) portfolio constraint that limits the deposit issuance of financial intermediaries—i.e., $b_{f,t} \geq 0$—and also their capital positions—i.e., $q_t k_{f,t} \geq 0$—according to

$$q_t k_{f,t} = n_{f,t} + b_{f,t} \leq \lambda V_t \ , \tag{6}$$

with the equality resulting from their balance sheets. The constraint always holds in equilibrium; otherwise, households would not be willing to hold deposits in the first place. Deposits are then de facto locally risk-free. Hereafter, I interpret deposit rate $r_t dt$ as the equilibrium risk-free real interest rate. A non-slack (i.e., relevant) portfolio constraint will be essential for the existence of systemic risk (see Section 4).

Portfolio Problems Portfolio optimization problems are standard. The problem of financial intermediaries is the same as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011), but cast in continuous time and with real investment decisions. The problem of households is the typical portfolio problem without portfolio constraint but with consumption and investment decisions. I lay out these problems in the next section, in which I solve the model.

Competitive Equilibrium In equilibrium, households and financial intermediaries optimize and markets for consumption and physical capital clear. The market for deposits automatically clears because of Walras' law.

3 Solving the Model

To solve the model, I proceed in three steps as follows. First, I lay out and solve the portfolio problems of financial intermediaries and households, in that order. Second, I combine optimality conditions with market clearing to obtain an aggregate production function and a marginal investor on physical capital in equilibrium. Lastly, I define the Markov competitive equilibrium, which is more a tractable—yet specific—equilibrium concept that I use to analytically characterize the solution to the model.

3.1 Portfolio Problem of Financial Intermediaries

Financial intermediaries pay out dividends to their owner household only once at an exogenous random time that occurs stochastically according to a Poisson arrival rate θ. The random times are idiosyncratic and i.i.d. across the intermediaries. When they pay out dividends, financial intermediaries transfer all of the accumulated net worth since inception, and immediately afterward, they are replaced by a newborn intermediary. The newborn receives a portion κ/θ of the aggregate capital stock from the same household as the initial endowment, with fraction $\kappa \in (0,1)$ being a parameter.[2]

Portfolio Problem The problem of financial intermediaries consists in maximizing the present discounted value of their dividend payout

$$V_t \equiv \max_{\iota_{f,t};k_{f,t}\geq 0} E_t \int_t^{\infty} \theta e^{-\theta(s-t)} \frac{\Lambda_s}{\Lambda_t} n_{f,s} ds \ , \tag{7}$$

subject to solvency constraint $n_{f,t} \geq 0$; the law of motion of net worth

$$dn_{f,t} = dR_{f,t} q_t k_{f,t} - \left(q_t k_{f,t} - n_{f,t}\right) r_t dt \ ; \tag{8}$$

and portfolio constraint (6). Financial intermediaries take the stochastic discount factor of households Λ_t as given. The intermediaries also take as given the price and rates of return in the problem.

Guess To solve the problem, I postulate that value function V_t is proportional to individual net worth $n_{f,t}$. That is:

$$V_t \equiv v_t n_{f,t} \ , \tag{9}$$

[2]This dividend payout scheme precludes financial intermediaries from saving away the portfolio constraint in equilibrium. Also, it precludes the intermediary sector from vanishing, as without net worth the intermediaries cannot borrow or operate. Lastly, the scheme allows a representative financial intermediary to exist in equilibrium (see below).

with $v_t \geq 1$ never being below 1, the same for all of the financial intermediaries, and independent of decisions $\iota_{f,t}$ and $k_{f,t}$. Putting the latter property differently, when making their investment and portfolio decisions, financial intermediaries take the already optimal marginal value of net worth v_t as given. In the model, however, v_t is endogenous, and will be determined later. This postulate implies that the market-based portfolio constraint is linear in individual net worth. In turn, this implies that the portfolio problem is scale invariant with respect to current individual net worth $n_{f,t}$. Optimality then implies that leverage multiple $q_t k_{f,t}/n_{f,t}$ is the same for all of the financial intermediaries regardless of their net worth. In equilibrium, a representative financial intermediary therefore exists.[3] To solve for optimal leverage multiple ϕ_t, I postulate that Λ_t and v_t evolve over time stochastically, according to diffusion processes with same shock dZ_t as in the law of motion of physical capital. I denote the drift and diffusion processes of any given variable x_t by $\mu_{x,t}$ and $\sigma_{x,t}$, respectively, in what follows. I derive optimal investment rate $\iota_{f,t}$, optimal leverage multiple ϕ_t, and the equilibrium condition for optimal marginal value of net worth v_t in the Online Appendix.

Investment Optimal investment rate $\iota_{f,t}$ maximizes rate of return $dR_{f,t}$. This determines a static optimization problem, whose solution is

$$I'\left(\iota_{f,t}\right) = \frac{1}{q_t} \ . \tag{10}$$

Leverage Optimal leverage multiple ϕ_t is

$$\phi_t = \left[\begin{array}{ll} \lambda v_t, & \text{if } \alpha_{f,t} > 0 \\ \beta_{f,t} & \text{if } \alpha_{f,t} = 0 \\ 0 & \text{if } \alpha_{f,t} < 0 \end{array}\right. , \tag{11}$$

with $\beta_{f,t}$ being a real number in interval $[0, \lambda v_t]$, and

$$\alpha_{f,t} \equiv \frac{1 - \iota_{f,t}}{q_t} + \mu_{q,t} + I\left(\iota_{f,t}\right) - \delta + \sigma_{q,t}\sigma - r_t + \left(\sigma_{q,t} + \sigma\right)\left(\sigma_{\Lambda,t} + \sigma_{v,t}\right) \tag{12}$$

the expected risk-adjusted excess return on physical capital over deposits that financial intermediaries earn during $(t, t + dt)$. If $\alpha_{f,t} > 0$, financial intermediaries strictly prefer physical capital to deposits. Thus, they take as much leverage as possible, and hit their leverage limit λv_t with ϕ_t. If $\alpha_{f,t} = 0$, financial intermediaries are indifferent between the two. Hence, they are also indifferent between any leverage multiple. Lastly, if $\alpha_{f,t} < 0$, financial intermediaries strictly prefer deposits and thus prefer not to hold physical capital. In equilibrium, only cases $\alpha_{f,t} > 0$ and $\alpha_{f,t} = 0$ are possible, however, and if $\alpha_{f,t} = 0$, then $\phi_t \geq 1$ cannot be below 1. This is because households do not issue deposits. Hereafter, I consider only those cases.

[3] See the Online Appendix for a formal derivation of the optimization problem and the representative financial intermediary. I also verify the postulate in the Online Appendix.

Tobin's Q In equilibrium, value v_t satisfies

$$\alpha_{f,t}\phi_t + \mu_{v,t} + \sigma_{\Lambda,t}\sigma_{v,t} + \frac{\theta}{v_t} - \theta = 0 \; , \tag{13}$$

with $\alpha_{f,t}\phi_t$ being the risk-adjusted excess return over deposits that financial intermediaries instead earn on net worth. This confirms that v_t measures the marginal value of net worth for financial intermediaries—i.e., the Tobin's Q. Because $\alpha_{f,t} \geq 0$ and $\phi_t \geq 1$, if $\alpha_{f,t} = 0$ always, then $v_t = 1$ always as well. This implies that in general, $v_t \geq 1$.[4]

3.2 Portfolio Problem of Households

Households have isoelastic preferences for consumption. Let $\gamma > 0$ denote the risk-aversion coefficient in their utility function. The stochastic discount factor of households is $\Lambda_t \equiv e^{-\rho t} c_t^{-\gamma}$, with $\rho > 0$ being their subjective time discount rate and c_t their consumption flows per unit of time.

Portfolio Problem The problem of households consists in maximizing the present discounted value of their utility flows

$$W_t \equiv \max_{\iota_{h,t}, k_{h,t}, c_t \geq 0} E_t \int_t^{\infty} e^{-\rho(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \; , \tag{14}$$

subject to solvency constraint $n_{h,t} \geq 0$, and the law of motion of their net worth,

$$dn_{h,t} = dR_{h,t} q_t k_{h,t} + (n_{h,t} - q_t k_{h,t})\, r_t dt - \tau_t dt - c_t dt \; , \tag{15}$$

with $k_{h,t} \geq 0$ being their positions on physical capital, $n_{h,t} - q_t k_{h,t}$ their savings in deposits, and τ_t the net transfers to financial intermediaries. In equilibrium, a representative household exists because individual households are identical. I solve the problem of households in the Online Appendix.

Solution Optimal investment rate $\iota_{h,t}$ maximizes $dR_{h,t}$. This specifies a static optimization problem as well, whose solution also is

$$I'(\iota_{h,t}) = \frac{1}{q_t} \; . \tag{16}$$

[4] I restrict attention to processes for v_t that are constant over time if $\alpha_{f,t}\phi_t$ is. This restriction ensures that if $\alpha_{f,t}\phi_t$ is a constant, both $\mu_{v,t} = 0$ and $\sigma_{v,t} = 0$ are null. If $\alpha_{f,t}\phi_t = 0$, then $v_t = 1$ always as well.

In equilibrium, investment rates $\iota_{h,t} = \iota_{f,t} = \iota_t$ then coincide. The optimal capital position of households is

$$q_t k_{h,t} = \begin{bmatrix} +\infty & \text{if } \alpha_{h,t} > 0 \\ \beta_{h,t} & \text{if } \alpha_{h,t} = 0 \\ 0 & \text{if } \alpha_{h,t} < 0 \end{bmatrix} , \tag{17}$$

with $\beta_{h,t} \in [0, +\infty)$, and

$$\alpha_{h,t} \equiv \frac{a_h - \iota_{h,t}}{q_t} + \mu_{q,t} + I(\iota_{h,t}) - \delta + \sigma_{q,t}\sigma - r_t + (\sigma_{q,t} + \sigma)\,\sigma_{\Lambda,t} \tag{18}$$

being the expected risk-adjusted excess return on physical capital over deposits that households earn during $(t, t+dt)$. The intuition for decision rule (48) is similar to that for rule (11). However, two important differences are that $\alpha_{h,t}$ does not depend on financial risk $(\sigma_{q,t} + \sigma)\,\sigma_{v,t}$ and $\alpha_{h,t} > 0$ cannot occur in equilibrium. These differences arise because households are not subject to portfolio constraints. Lastly, at the optimal, households match their expected utility return from consumption to the real interest rate. This implies that

$$-\mu_{\Lambda,t} \equiv \rho + \gamma\mu_{c,t} - \frac{1}{2}\gamma(\gamma+1)\,\sigma_{c,t}^2 = r_t \ , \tag{19}$$

with $\mu_{c,t}$ and $\sigma_{c,t}$ being the drift and diffusion processes of consumption, respectively.

3.3 Aggregate Production Function and a Marginal Investor

An aggregate production function and a marginal investor on physical capital exist in equilibrium. These objects are critical for the characterization of the Markov equilibrium. Because a representative household and a representative financial intermediary also exist, in what follows, to economize on notation, I make no distinction between individual and aggregate variables.

Aggregate Production The aggregate production function is the result of aggregating the individual capital positions of financial intermediaries and households. The function is

$$y_t = [\phi_t\eta_t + a_h(1 - \phi_t\eta_t)]\,k_t \ , \tag{20}$$

with $\eta_t \equiv n_{f,t}/(n_{h,t} + n_{f,t}) = n_{f,t}/q_t k_t \in [0,1]$ being the aggregate net worth of financial intermediaries as a share of total wealth, and $n_{h,t} + n_{f,t} = q_t k_t$ because physical capital is the single real asset. In equilibrium, $\phi_t\eta_t = k_{f,t}/k_t \in [0,1]$ equals the aggregate capital share of financial intermediaries, and the term in brackets is the aggregate productivity of physical capital, which I denote by $\zeta_t \equiv \phi_t\eta_t + a_h(1-\phi_t\eta_t) \in [a_h, 1]$. Variables η_t and k_t will be the two states in the Markov equilibrium.

Marginal Investor Decision rules (11) and (48), coupled with market clearing, imply that $\alpha_{h,t} \leq 0 \leq \alpha_{f,t}$, with either one of the two weak inequalities always holding with equality;

otherwise, no marginal investor on physical capital would exist. I postulate that both weak inequalities never hold with equality simultaneously. Then

$$\begin{cases} \alpha_{f,t} = 0 > \alpha_{h,t} & \text{when } \lambda v_t \eta_t \geq 1 \\ \alpha_{h,t} = 0 < \alpha_{f,t} & \text{otherwise} \end{cases} , \tag{21}$$

and $\phi_t = \min\{\lambda v_t, 1/\eta_t\}$. This implies that financial intermediaries are the marginal investor only when they, as a whole, have sufficient borrowing capacity to hold all of the aggregate capital stock. In that case, $k_{f,t}/k_t = \phi_t \eta_t = 1$, because $\alpha_{h,t} < 0$. The expression also implies that households are the marginal investor otherwise. In this other case, $\phi_t \eta_t = \lambda v_t \eta_t < 1$ is feasible, because $\alpha_{h,t} = 0$. Condition (21) can be regarded as the equilibrium pricing equation for physical capital.

3.4 Markov Competitive Equilibrium

A Markov competitive equilibrium is a set of state variables Γ and a set of mappings $x : \Gamma \to \Gamma^c$ such that endogenous variables in Γ evolve in accord, and mappings $x : \Gamma \to \Gamma^c$ are consistent with the conditions of the competitive equilibrium. I conjecture that a Markov equilibrium exists. Also, I conjecture that the state variables in the equilibrium are $\Gamma = \{\eta, k\}$ and mappings x are either linear on or independent of state k. The additional conjectures render the size of the economy proportional to the aggregate capital stock. I thus interpret k as the economic trend. Hereafter, because I restrict attention to Markov competitive equilibria, I omit time subscript $_t$ when denoting variables.

Equilibrium Conditions The conditions of the Markov equilibrium are the following: The leverage multiple is $\phi = \min\{\lambda v, 1/\eta\}$; aggregate consumption is $c = (\zeta - \iota)\, k$; aggregate productivity is $\zeta = \phi\eta + (1 - \phi\eta)\, a_h$; investment rate ι satisfies $I'(\iota) = 1/q$; Tobin's Q v satisfies pricing equation (39); real deposit rate r satisfies pricing equation (19); the price of physical capital q satisfies pricing equation (21); and aggregate capital stock k and wealth share η evolve over time stochastically according to (2) and (22), respectively, with

$$\frac{d\eta}{\eta} = \mu_\eta dt + \sigma_\eta dZ , \tag{22}$$

with[5]

$$\mu_\eta \equiv \frac{1-\iota}{q}\phi + \left[\mu_q + I(\iota) - \delta + \sigma_q\sigma - r - (\sigma_q + \sigma)^2\right](\phi - 1) - \left(\theta - \frac{\kappa}{\eta}\right) , \tag{23}$$

$$\sigma_\eta \equiv (\phi - 1)(\sigma_q + \sigma) . \tag{24}$$

[5] This law of motion follows from, first, applying Ito's quotient rule to $\eta = n_f/qk$, and then subtracting from the resulting expression the net transfers from financial intermediaries to households, $\theta - \kappa/\eta$.

Solution Method To solve the Markov equilibrium, it suffices to solve for mappings $\{q, v\}$. This is because any other mapping can be derived from these two. Mappings $\{q, v\}$ can be analytically characterized as the solution to a second-order ordinary differential equation system (ODEs) in state η. The ODEs is given by pricing equations (39) and (21).[6] I derive the ODEs in the Online Appendix and solve it numerically using spectral methods and the parameter values in Table 1.

4 Systemic Risk and the Natural Rate

To examine the relationship between systemic risk and the natural rate, I proceed progressively in two steps. First, I consider an economy without financial frictions. This economy is suited for analyzing the effect of exogenous aggregate consumption risk on the natural rate. Second, I consider an economy with the frictions. This other economy instead has systemic risk and endogenous risk in aggregate consumption. Therefore, interactions between those types of risks and the natural rate exists. I interpret the two economies as having nominal rigidities in price setting—with the rigidities being the same as those in the economies in Section 5—; a passive monetary policy that is concerned only with macroeconomic stabilization in the short term, and either a slack or no ZLB constraint on nominal interest rates. Because economies with those characteristics attain macroeconomic stabilization, in this section, I interpret rdt as the natural rate.

4.1 A Frictionless Economy

In this economy, financial contracts are enforceable and financial markets are complete. Perfect enforcement renders defaulting impossible. Put formally, $\lambda = +\infty$, which rules out the market-based portfolio constraint. Financial market completeness allows intermediary net worth to be negative, in which case I say that financial intermediaries issue equity. These two features combined guarantee allocative efficiency. In particular, intermediary capital share $k_f/k = \phi\eta = 1$ equals 1, as do aggregate productivity $\zeta = 1$ and output-to-capital ratio $y/k = 1$. Value $v = 1$ also equals 1 and excess return $\alpha_f = 0 < \alpha_h$ is null. The price of physical capital q, investment rate ι, and natural rate r jointly solve

$$\frac{1-\iota_E}{q_E} + [I(\iota_E) - \delta] - r_E = \gamma\sigma^2 \; ; \quad I'(\iota_E) = \frac{1}{q_E} \; ; \quad r_E = \rho + \gamma [I(\iota_E) - \delta] - \frac{1}{2}\gamma(\gamma+1)\sigma^2 \; ; \tag{25}$$

with the subindex $_E$ denoting efficient values.[7]

The above equations show that the efficient natural rate is determined exclusively by preferences and technology. The efficient rate is strictly increasing in the time discount rate and the

[6] Recall that Ito's Lemma allows expressing drift and diffusion processes as a function of first- or second-order derivatives of the underlying variable with respect to the state.

[7] The first equation follows from (21). The second follows from (10) and (16). The third follows from (1), (2), and (19). See the Online Appendix for details.

expected growth rate of aggregate consumption. It is instead strictly decreasing in the volatility of the consumption growth rate. This last result holds because in equilibrium, the insurance value of risk-free claims increases with aggregate consumption risk. Put differently, because safe assets are valuable as a hedge against systematic risk, their rate of interest falls as that source of risk increases. All else equal, the efficient natural rate is lower the riskier the technological fundamentals or the higher the aversion to risk. The efficient rate will be critical for determining the level around which the natural rate fluctuates in the economy with frictions.

4.2 The Frictional Economy

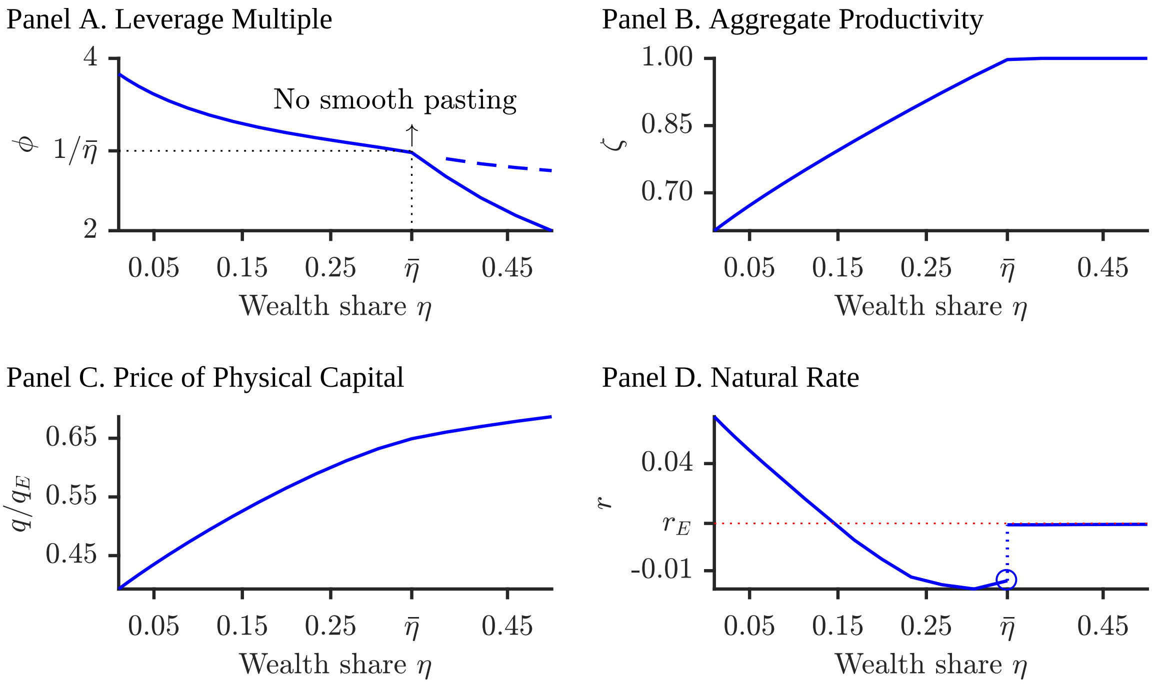
In this other economy, the market-based portfolio constraint is $\phi \leq \lambda v$. I postulate that in equilibrium, value v is bounded from above and strictly decreasing in η, which implies that the constraint is binding only when the wealth share is sufficiently low. Put formally, there exists a threshold state $\bar{\eta} \in (0,1)$, with $\lambda v(\bar{\eta})\bar{\eta} = 1$, such that $\phi = \lambda v < 1/\eta$ when $\eta < \bar{\eta}$ and $\phi = 1/\eta < \lambda v$ when $\eta > \bar{\eta}$. The pricing condition for physical capital in the ODEs then reduces to $\alpha_h = 0 < \alpha_f$ when $\eta < \bar{\eta}$, while $\alpha_f = 0 > \alpha_h$ when $\eta \geq \bar{\eta}$.

I solve the reduced ODEs numerically. Figure 1 plots the Markov competitive equilibrium as a function of the wealth share. Figure 2 plots the dynamics of the wealth share in the top panels and aggregate output risk in the bottom panel. The dynamics are characterized by the law of motion and the probability distribution in the long run (i.e., the invariant distribution) of the wealth share.

Figure 1 shows that the equilibrium has two well-demarcated regions. These regions differ inherently on whether financial intermediaries as a whole have or lack enough borrowing capacity to hold all of the aggregate capital stock. When $\eta \geq \bar{\eta}$ is high, financial intermediaries have such a borrowing capacity, collectively hold all of the aggregate capital stock, and their leverage constraint $\phi = 1/\eta \leq \lambda v$ is slack (Figure 1A). The allocation of physical capital and aggregate productivity are efficient (Figure 1B). However, the price of physical capital and the investment rate are not (Figure 1C). When $\eta < \bar{\eta}$ is instead low, financial intermediaries lack such a borrowing capacity, collectively hold as much physical capital as possible, and their leverage constraint $\phi = \lambda v < 1/\eta$ is binding. Aggregate productivity $\zeta < 1$ is inefficient, as are the price of physical capital and the investment rate. When compared with the other region, however, investment is even more inefficient, because households are the marginal investors in this region and financial intermediaries are in the other.

Figure 2 shows that the wealth share oscillates continuously between the two regions (Figures 2A or 2B, histogram). The fluctuations of the wealth share are stochastic (generically), because diffusion process $\sigma_\eta > 0$ is positive (Figure 2B, solid line), which follows from the difference in riskiness between rates of return dR_f and rdt. The volatility of the fluctuations is nonlinear. In particular, the volatility is inversely U-shaped related to the wealth share and it peaks precisely when the market-based limit on leverage is locally occasionally binding (that is, close below $\eta = \bar{\eta}$). In that region, financial intermediaries collectively are sufficiently well capitalized to

Figure 1: Markov Equilibrium as a Function of the Wealth Share



Notes: The dashed line in Panel A plots the market-based limit on leverage (i.e., λv). The leverage multiple is $\phi = \min\{\lambda v, 1/\eta\}$.

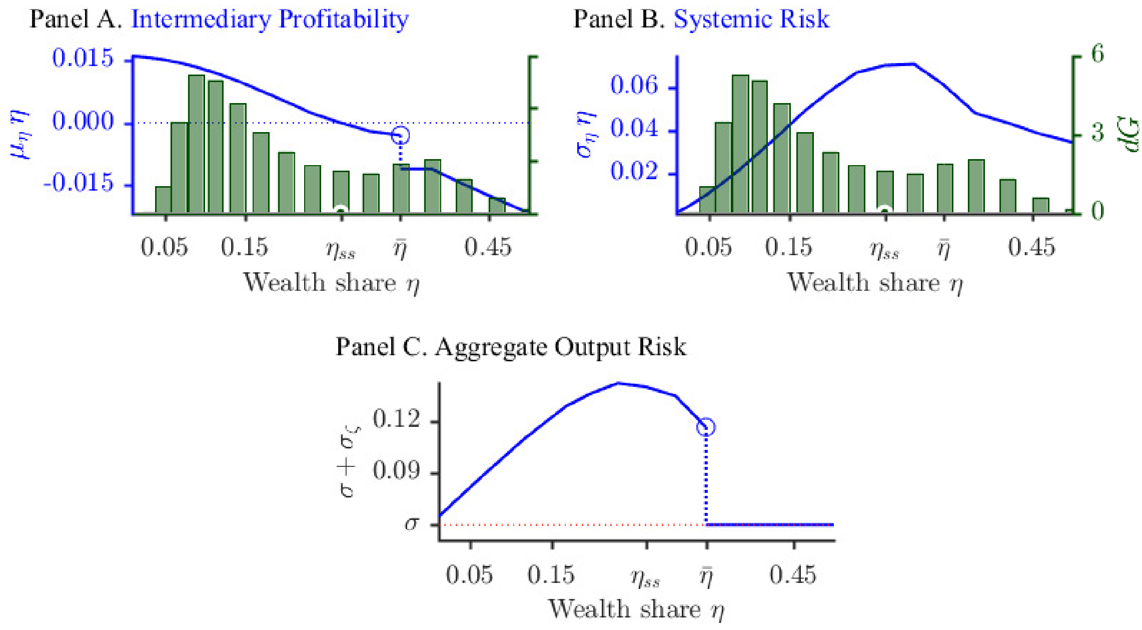
exert large aggregate effects, but not to tolerate adverse shocks $dZ < 0$ without selling physical capital at discount prices to households. The fluctuations of the wealth share are also mean-reverting (Figure 2A, solid line). Formally, the wealth share tends to revert in expectation to its stochastic steady state, $\eta = \eta_{ss}$, with $\eta_{ss} < \bar{\eta}$ being such that $\mu_\eta(\eta_{ss}) = 0$. Mean reversion occurs because interest-rate margins and intermediary profitability are high when households are the marginal investors but low when the intermediaries are.

Over a sufficiently long time horizon, the fluctuations shape a stochastic cycle that oscillates recurrently around the trend throughout, from booms to busts (Figures 2A and 2B). Booms refer to episodes in which financial conditions are sound—that is, $\eta \geq \bar{\eta}$—and aggregate productivity is efficient. Busts instead refer to episodes in which the conditions are extremely tight—that is, $\eta \ll \bar{\eta}$—and aggregate productivity is extremely inefficient. Investment and the expected growth trend also fluctuate over the cycle. In particular, both ι and $I(\iota) - \delta$ are procyclical, that is, positively related to the wealth share/financial conditions.[8]

The Natural Rate The natural rate also fluctuates over the cycle (Figure 1D). As a function of the wealth share, the rate is U-shaped up to the threshold, jumps upward at the threshold,

[8]The behavior of the wealth share is also key for explaining the level of forward-looking variables such as the price of physical capital and the Tobin's Q. Specifically, $q < q_E$ is always below efficiency and $v > 1$ is always above, because marginal investors are forward-looking and because the cycle oscillates continuously from busts to booms.

Figure 2: Dynamics of the Wealth Share and Aggregate Output Risk



Notes: The invariant density function satisfies $\ln dG = -2\ln(\sigma_\eta \eta) + 2\int_0^\eta \mu_{\tilde{\eta}}\tilde{\eta}/(\sigma_{\tilde{\eta}}\tilde{\eta})^2 \, d\tilde{\eta} + constant$. Detrended aggregate output risk satisfies $\sigma_\zeta = \varepsilon_\zeta \sigma_\eta$, with $\varepsilon_\zeta \equiv (\partial\zeta/\partial\eta)(\eta/\zeta)$.

and remains relatively constant beyond. To understand the behavior, it is useful to decompose the rate in five terms as follows:

$$r = r_E + \gamma\left[I(\iota) - I(\iota_E)\right] + \gamma\mu_{c/k} - \frac{1}{2}\gamma(\gamma+1)\left(\sigma_{c/k} + 2\sigma\right)\sigma_{c/k} + \gamma\sigma_{c/k}\sigma \ , \qquad (26)$$

with $\mu_{c/k}$ and $\sigma_{c/k}$ being the drift and diffusion processes of detrended consumption $c/k = \zeta - \iota$, respectively.[9] The first term on the RHS is the efficient natural rate. The remaining terms are expressed as deviations from efficiency. These terms can be interpreted as the many channels through which the frictions affect the natural rate. The second term follows from the deviation in the expected growth trend. It captures the pressures that investment distortions exert on the rate. The last three terms follow from the deviations in the growth rate of detrended consumption. The sum of the last two terms captures the combined pressures from systemic risk and endogenous aggregate consumption risk. I refer to the sum as the risk premium channel in what follows.[10]

The second term in the decomposition is continuous, negative, and strictly increasing. It is continuous because physical capital is traded continuously—were the price of physical capital instead to jump, arbitrage opportunities would exist. The two other properties of the term follow from the price's being inefficient and procyclical. Distortions in investment thus exerts

[9] The decomposition follows Ito's product rule.

[10] Note that for any generic variable x, $\sigma_x = \varepsilon_x \sigma_\eta$, with $\sigma_x = \varepsilon_x \equiv (\partial x/\partial\eta)(\eta/x)$. This follows from Ito's Lemma.

downward pressure on the natural rate continuously throughout the cycle. The tighter the financial conditions, the higher the pressures.

The remaining terms are nonmonotonous and jump at the threshold. Up to the threshold the third term is strictly decreasing. This is because of the mean-reverting behavior of the wealth share, which implies that in expectation, aggregate productivity recovers during busts but declines close below the booms. The risk premium channel is U-shaped. Its behavior follows primarily from the behavior of (detrended) aggregate output risk $\sigma_\zeta = \varepsilon_\zeta \sigma_\eta$, with $\varepsilon_x \equiv (\partial x/\partial \eta)(\eta/x)$ being the elasticity of generic variable x with respect to the wealth share, and $\sigma_x = \varepsilon_x \sigma_\eta$ because of Ito's Lemma. Output risk is inversely U-shaped (Figure 2C) because systemic risk $\sigma_\eta \eta$ is.

The discontinuity at the threshold contributes to the nonmonotonicity. The discontinuity arises generally because no equilibrium condition imposes smooth pasting between λv and $1/\eta$ at $\eta = \bar{\eta}$. (Figure 1A).[11] The resulting kink in the leverage multiple generates jumps, not in the level, but in the expected rate of change and the volatility of the rate of change of aggregate productivity. Put formally, $\zeta = \phi\eta + a_h (1 - \phi\eta)$ is continuous, but $\partial\zeta/\partial\eta$ and $\partial^2\zeta/(\partial\eta)^2$ are not—which renders μ_ζ and σ_ζ discontinuous as well. Processes $\mu_{c/k}$ and $\sigma_{c/k}$ thus are also discontinuous.

At the threshold, both the third term and the risk premium channel jump upward to a number close to zero, and beyond the threshold, they remain close to that number. The jump is not exactly to zero, because of the rate of change of investment, which negligibly affects the terms. For the same reason, the terms do not remain fully stable at zero beyond.[12]

All in all, fluctuations in aggregate productivity to a major extent, and the behavior of investment to a minor extent, explain almost all of the fluctuations in the natural rate. The natural rate fluctuates the most violently when systemic risk is at its highest levels. The rate also tanks during such turbulent phases. Systemic risk is indeed a key determinant of the lower bound on the rate.

4.2.1 Comparative Statics

This subsection examines the role of key parameters in the cyclical behavior of the natural rate. It also disentangles the contribution of the terms in the decomposition to the lower bound of the rate. The comparative static analysis uses the baseline parameter values as the benchmark. Table 1 displays the values. The time frequency is annual.

The baseline abstracts away from depreciation in physical capital. That is, $\delta = 0\%$. The return on investment is $\ln I(\iota) = \ln I_0 + I_1 \ln \iota$, with $I_0 \geq 0$ and I_1 being two parameters. Values

[11] The boundary conditions of the ODEs are $\sigma_v \to 0$, $\partial\sigma_v/\partial\eta \to 0$, $\sigma_q \to 0$, and $\partial\sigma_q/\partial\eta \to 0$, as $\eta \to 1$. These conditions require risk quantities σ_v and σ_q to vanish smoothly as the intermediaries collectively own the entire wealth. See the Online Appendix for details. There is no reason in the setup for the market-based limit on and the efficient quantity of leverage to satisfy smooth pasting at the threshold in equilibrium.

[12] In the stagnant economies that I consider, the law of motion of invesment (i.e., $d\iota/\iota = \mu_\iota dt + \sigma_\iota dZ$) has only negligible effects on the natural rate. This is because the level of investment is already small relative to the level of output. See the below subsection for comparative statics.

Table 1 — Baseline Parameter Values

Parameter	Expression	Value
Panel A. Investment		
Depreciation rate	δ	0%
Constant in log investment return function	$\ln I_0$	−11.51
Slope in the log return function	I_1	−2.20
Panel B. Financial Intermediation		
Intermediation edge	$1-a_h$	40%
Fraction of divertible assets	$1/\lambda$	40%
Frequency of dividend payouts	$1/\theta$	10
Initial endowment of starting intermediary	κ/θ	15%
Panel C. Preferences and Exogenous Risk		
Subjective time discount rate	ρ	1.5%
Risk-aversion coefficient	γ	1.5
Exogenous systematic risk	σ	6.0%

$\ln I_0 = -11.51$ and $I_1 = -2.2$ are consistent with an unconditional investment-to-output ratio and natural rate of $\int \iota(\eta)/\zeta(\eta)\,dG(\eta) = 2.88\%$ and $\int r(\eta)\,dG(\eta) = 0.73\%$, respectively, with G being the invariant distribution. All of the other parameters fixed, lower values of any of those two parameters reduce both unconditional means; thus, they generate more stagnant economies with secularly lower growth trends and natural rates. In the limit—i.e., if $I_0 \to 0$—investment vanishes, and fluctuations in aggregate productivity alone drive the behavior of the natural rate.

Parameter values $a_h = 60\%$, $\lambda = 2.5$, $\theta = 10\%$, and $\kappa = 1.5\%$ are consistent with an unconditional Sharpe ratio, leverage multiple, intermediary wealth share, and frequency of dividend payouts of $\int SR(\eta)\,dG(\eta) = 24\%$, $\int \phi(\eta)\,dG(\eta) = 3.02$, $\int \eta dG(\eta) = 24\%$, and $1/\theta = 10$, respectively, with $SR \equiv [E[dR_f/dt|\eta] - r]/(\sigma_q + \sigma)$ being the conditional Sharpe ratio. A larger intermediation edge $1 - a_h > 40\%$ increases the valuation difference on physical capital between financial intermediaries and households. A larger valuation difference intensifies the discontinuities in the endogenous risk quantities and therefore increases systemic risk. The natural rate is then more unstable, its lower bound is even lower, and the risk premium channel has a larger impact on the fluctuations of and the lower bound on the rate (Table 2).

A (marginal) smaller share of divertable assets $1/\lambda \lesssim 40\%$ allows the intermediaries to take more risk. Its effects on systemic risk and the natural rate, therefore, are similar to those of the larger intermediation edge.

The frequency of dividend payouts and the starting endowment affect systemic risk and the natural rate primarily via the expected recovery pace of intermediary net worth. A higher frequency $1/\theta > 10$ or a lower endowment $\kappa/\theta < 15\%$ slow recapitalization and hence increase systemic risk while reducing the rate.

Lastly, parameter values $\rho = 1.5\%$, $\gamma = 1.5$, and $\sigma = 6\%$ pin down the subjective time discount rate, aversion to risk, and exogenous systematic risk, respectively. The discount rate affects the natural rate primarily via the efficient rate. A larger risk-aversion coefficient $\gamma > 1.5$ increases the relative importance of the risk premium channel in the rate decomposition and thus depresses the natural rate. As opposed to the "volatility paradox" result in Brunnermeier and Sannikov (2014), a lower risk quantity $\sigma < 6\%$ increases neither systemic risk nor the endogenous risk quantities. This is because the leverage constraint—which is not present in that paper—precludes financial intermediaries from disproportionately increasing risk-taking as exogenous risk diminishes. Put simply, even though concentration of aggregate risk in the balance sheets of the intermediaries also increases, it does not do so to the extent of boosting endogenous risk beyond the reduction in exogenous risk.

Table 2 — Effect of Key Parameters on Natural Rate

	Frequencies, moments, and values			
	$\Pr[r<0]$	$E[r \mid r<0]$	$\min r$	$\sigma_{c/k}/\min r$
Panel A. Baseline parameter values	33%	−1.19%	−1.59%	66%
Panel B. Comparative statics				
No investment (i.e., $I_0 = 0$)	36%	−1.58%	−2.27%	70%
Softer leverage constraint (i.e., $1/\lambda = 35\%$)	28%	−2.34%	−3.47%	76%
Higher aversion toward risk (i.e., $\gamma = 2$)	40%	−3.37%	−5.79%	73%
Higher exogenous risk (i.e., $\sigma = 10\%$)	57%	−2.21%	−4.61%	82%

Notes: The parameter values in the comparative statics exercises are the same as in the baseline, except for the parameters noted in the table. Expression "$\sigma_{c/k}/\min r$" denotes the contribution of the risk premium channel to the lower bound on the natural rate. This contribution is computed as the ratio of the channel to $r - r_E$ at the state point at which $r = \min r$.

5 Liquidity Traps and Macro-prudential Policy

So far, the analysis has abstracted away from the possibility of liquidity traps and from monetary and macro-prudential policy intervention. To integrate the traps and monetary policy in the setup, I follow the approach of Caballero and Simsek (2020). Essentially, the model now is a New Keynesian economy with fully rigid nominal prices, in which aggregate output can fall below installed capacity in equilibrium. The monetary instrument is the short-term nominal interest rate, as usual, but because the inflation rate is null, monetary policy sets the counterpart real interest rate directly. Central to the analysis is that neither can the policy rate nor its counterpart real rate fall below zero in equilibrium. This ZLB constraint exists because cash-like assets—which for simplicity are in zero-net supply in the setup—preclude nominal rates from being negative. As the macro-prudential intervention, I consider an additional, policy-based limit on leverage, which for simplicity is the same for all of the intermediaries. This limit is state-contingent meaning that it can restrict intermediary leverage further below the market-based limit occasionally—depending on the state—with varying degrees of intensity when it does so. I restrict attention to these two instruments because they are the most standard monetary and macro-prudential policies, respectively, in actual economies.

5.1 The Model with the ZLB Constraint

In what follows, I detail the added features while simultaneously I solve the resulting model. For convenience, I begin the exposition using time-$_t$ notation. To finish solving the model, I consider the Markov equilibrium. Any feature of the setup I do not explicitly mention in this section remains the same as in Section 2.[13]

Technology A continuum of identical firms each produce a single differentiated good flow $c_{j,t}$ per unit of time, with $j \in [0,1]$, using as input the gross output flows that physical capital delivers, that is, $y_t = \zeta_t k_t$. The production technology is one-to-one. The differentiated goods are the inputs in the production of the consumption good. This production is done via a CES aggregator

$$c_t = \left[\int_0^1 c_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} , \tag{27}$$

with parameter $\varepsilon > 1$ being the (constant) elasticity of substitution.

Nominal Rigidities The nominal price of the differentiated goods is preset and firms cannot reset the price. This is the source of nominal rigidities in the model. The price is the same among the goods. These features combined imply that the price in terms of consumption equals 1 in equilibrium. Formally, $p_{j,t}/p_t = p_j/p = 1 \ \forall j \in [0,1]$, with $p_{j,t} = p_j = p$ being the nominal price of the differentiated goods and $p_t = \left[\int_0^1 p_{j,t}^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}}$ the nominal price of consumption. As usual in the New Keynesian framework, the CES aggregator minimizes production costs while taking input prices as given—which explains the equilibrium expression for p_t—and the equilibrium is scale invariant with respect to the nominal price level.

Demand-driven Equilibrium The CES aggregator also determines a demand system for the differentiated goods. The demand functions in the system are $c_{d,t}(p_{j,t}) \equiv (p_{j,t}/p_t)^{-\varepsilon} c_t = c_t$ $\forall j \in [0,1]$, with the equality holding only in equilibrium. To supply their indirect demand, firms purchase the output flows in competitive markets. Because the production technology is one-to-one, the real price of the output flows also equals 1, which ensures that firms break even regardless of the quantity sold. In what follows, I restrict attention to symmetric equilibria with the possibility of rationing. In these equilibria, all of the firms behave in the same manner, and as a whole, they can use less in production than the aggregate quantity of the output flows. Aggregate output can then fall below production capacity, in which case aggregate consumption

[13]The framework of Caballero and Simsek (2020) is suited for this application because it does not add state variables to the Markov equilibrium beyond those in the economy with flexible prices (Section 2). See Van der Ghote (2020) for a more general New Keynesian economy in continuous time with a flexible degree of nominal price rigidity. In that economy, dispersion in intermediate goods prices is also a state in the Markov equilibrium.

and investment alone determine the equilibrium output. Let $u_t \equiv (c_t + \iota_t k_t)/y_t \in [0, 1]$ denote the capacity utilization rate. In the baseline model, $u_t = 1$ necessarily, but in this model, $u_t < 1$ is possible. The utilization rate isolates the potential output losses from nominal rigidities. This is because production capacity $y_t = \zeta_t k_t \leq k_t$ already incorporates the potential losses from the financial frictions. To close the model, I must specify the monetary policy and macro-prudential policy in place.

Monetary Policy I consider a passive monetary policy rule that mimics the natural rate whenever possible and remains stuck at the ZLB otherwise. Formally, $i_t = \max\{r_t, 0\}$, with $i_t \geq 0$ being the nominal interest rate and r_t the natural rate of return. In equilibrium, nominal and real interest rates coincide because the inflation rate is null. The natural rate is such that $r_t = -\mu_{\bar{\Lambda},t}$, with $\bar{\Lambda}_t \equiv e^{-\rho t}\left[(\zeta_t - \iota_t) k_t\right]^{-\gamma}$. Put differently, r_t is the equilibrium interest rate that is consistent with full capacity utilization. By definition, if $r_t < 0$, then $i_t = 0$ and $u_t < 1$, which means that an artificially high real interest rate, a subdued aggregate demand, and stagnation in economic activity ensue. I interpret such an event as a liquidity trap. The passive rule can be regarded as a monetary policy that is concerned only with macroeconomic stabilization in the short term.

Macro-prudential Policy I consider a policy-based limit that can be set contingent only on the wealth share. That is, $\Phi_t \equiv \Phi(\eta_t)$, with $\Phi_t \geq 1$ being the limit and $\Phi(.)$ a mapping whose image cannot be below 1. This restriction ensures that variables $\{\eta_t, k_t\}$ summarize the state of the economy and that the economy's size is proportional to the aggregate capital stock. These properties hold because the policy-based limit directly affects the borrowing capacity of the intermediaries alone.[14] The restriction can be interpreted as a standard macro-prudential policy that can respond only to fluctuations in financial conditions.

Equilibrium Conditions The competitive equilibrium takes the policy rules as given. I restrict attention to the Markov equilibrium in what follows. The conditions that analytically characterize the Markov equilibrium are the same as in the baseline model except for the following three features. First, gross dividend returns on physical capital are deflated by the utilization rate. Formally, the returns are $uak \leq ak$. All else equal, then, financial intermediaries earn less on physical capital during than outside the liquidity trap. Second, households match their expected marginal utility return from consumption to the real policy rate. That is, $-\mu_\Lambda = i \geq r = -\mu_{\bar{\Lambda}}$, with $\Lambda_t \equiv e^{-\rho t}\left[u_t (\zeta_t - \iota_t) k_t\right]^{-\gamma}$. During the liquidity trap, therefore, financial intermediaries face higher borrowing costs than the ones they would face were nominal

[14]The portfolio problem of financial intermediaries is the same as in the baseline model but with an additional portfolio constraint, $q_t k_{f,t}/n_{f,t} \leq \Phi_t$. Financial intermediaries also take Φ_t as given. These properties ensure that the solution to the problem is also the same as in the baseline, but with an effective leverage limit of $\min\{\lambda v_t, \Phi_t\}$ rather than of λv_t. Notably, the two solutions coincide, if $\Phi_t = +\infty$.

prices flexible. Third, the leverage multiple is $\phi = \min\{\lambda v, \Phi, 1/\eta\}$, with households being the marginal investors when $\min\{\lambda v, \Phi\}\eta < 1$ and the intermediaries being otherwise.

Solution Method An ODEs in η for $\{q, v, u\}$ analytically characterize the solution to the model. Outside the trap, the ODEs is given by the pricing equations for physical capital and intermediary net worth alone—as in the baseline model. The capacity utilization rate is $u = 1$. The policy and the natural rate coincide and satisfy $i = -\mu_\Lambda = r = -\mu_{\bar{\Lambda}} \geq 0$. Inside the trap, the ODEs is given by those same equations plus $-\mu_\Lambda = 0$, with $u < 1$ and $i = -\mu_\Lambda = 0 > r = -\mu_{\bar{\Lambda}}$. The additional equation specifies an ODE for capacity utilization. During the trap, the natural rate is a counterfactual rate that does not affect the equilibrium.

5.2 A Financially Unregulated Economy

First, I abstract away from macro-prudential intervention. That is, I set $\Phi = +\infty$. I proceed in this way to first isolate the effects of the passive monetary policy rule and the ZLB constraint on the relationship between systemic risk and the natural rate. I consider the same parameter values as in Table 1. Under those values, the fundamentals of the economy are weak, but not too weak, in the following sense: In the economy without the constraint (see Section 4), large spikes in systemic risk push the natural rate down into negative terrain. However, once the risk falls back to normal levels, the rate recovers and re-enters the positive terrain. Formally,

$$\lim_{\eta \to \bar{\eta}^-} r(\eta) < 0 < \lim_{\eta \to \bar{\eta}^+} r(\eta) \, , \tag{28}$$

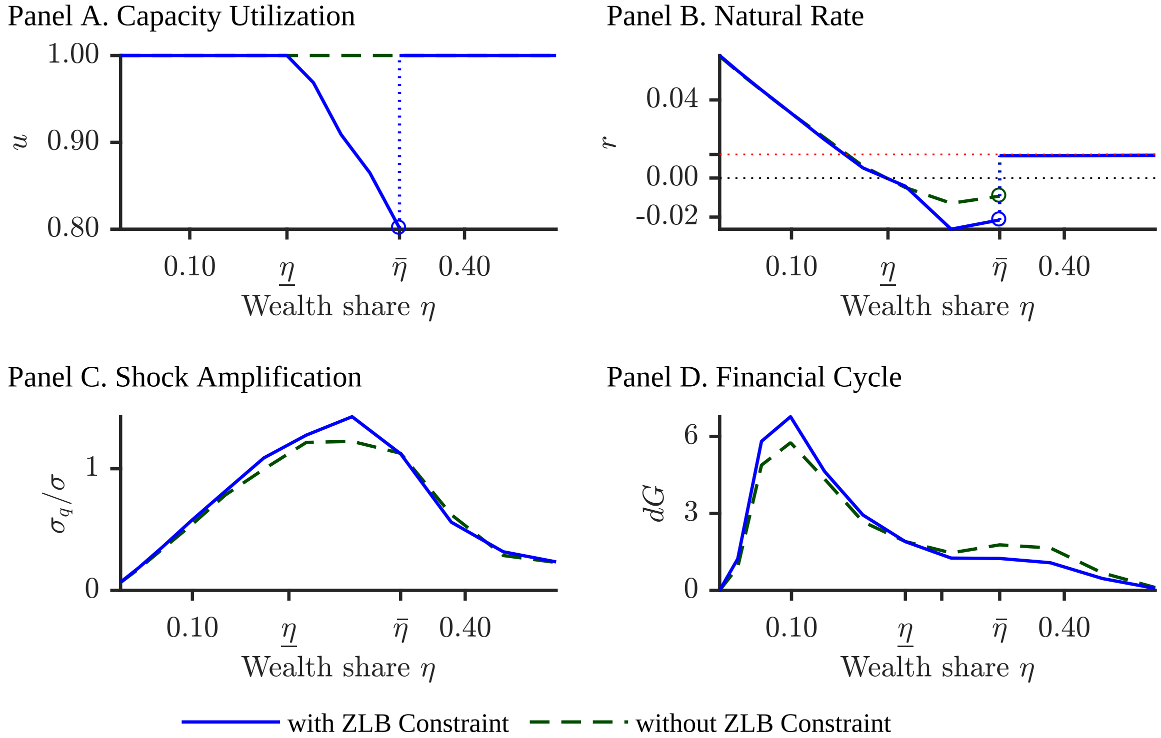
with the objects in the expression denoting the corresponding objects in the economy without the constraint. In economies whose fundamentals satisfy the condition, the ZLB constraint is slack during and occasionally binding below the booms. I restrict attention to this class of occasionally binding ZLB constraints in what follows.

To solve the model, I postulate the same properties on the Tobin's Q as in Subsection 4.2. I solve the reduced ODEs numerically as well. Figure 3 plots the Markov equilibrium as a function of the wealth share and the invariant distribution of the wealth share. The figure also plots the same objects for the economy without the constraint.

The ZLB constraint is binding when systemic risk navigates around its peak (Figures 3B and 3C). This happens from right below the booms to phases not too far into the busts. Formally, $i = 0 > r$ only when $\eta \in (\underline{\eta}, \bar{\eta})$, with the lower threshold being such that $r(\underline{\eta}) = 0$ and the upper threshold defined in the same manner as in the economy without the constraint. During the trap, the natural rate is negative, the policy rate is stuck at the ZLB, aggregate demand is subdued, and aggregate output is below installed capacity (Figures 3A and 3B), in effect.

During the trap, capacity underutilization fluctuates and it does so in a manner that is consistent with a null expected marginal utility return from consumption. The fluctuations (i.e., $du/u = \mu_u dt + \sigma_u dZ$) affect the utility return directly via two channels. First, the stochastic

Figure 3: Equilibrium with an Occasionally Binding ZLB Constraint



Notes: The shock amplification factor satisfies $\sigma_q/\sigma = (\phi-1)\,\varepsilon_q/\left[1-(\phi-1)\,\varepsilon_q\right]$.

component—that is, $\sigma_u dZ$—affects both detrended output risk $\sigma_{c/k+\iota} = \sigma_u + \sigma_\zeta$ and detrended covariance $\sigma_u\sigma_\zeta$. Notably, a downward-sloping rate renders capacity utilization negatively correlated with production capacity and hence reduces both risk quantities. Formally, after detrending, $\sigma_u = \varepsilon_u\sigma_\eta < 0$ becomes negative while $\sigma_\zeta = \varepsilon_\zeta\sigma_\eta > 0$ remains positive. Second, the deterministic component—that is, $\mu_u dt$—affects expected detrended output growth rate $\mu_{c/k+\iota} = \mu_u + \mu_\zeta + \sigma_u\sigma_\zeta$. Notably, again, a downward-sloping utilization rate boosts the output growth rate when $\eta > \eta_{ss}$, whereas it reduces that same rate when $\eta < \eta_{ss}$. This is because of the mean-reverting behavior of the wealth share at the stochastic steady state. In equilibrium, the utilization rate is downward-sloping (Figure 3A), indeed. When $\eta > \eta_{ss}$, the utilization rate exerts upward pressures on the utility return via both channels. When $\eta < \eta_{ss}$, the rate instead does so only via the risk channel. In both regions, the upward pressures are just strong enough to lift actual utility return $-\mu_\Lambda = 0$ above its counterfactual counterpart $-\mu_{\bar{\Lambda}} < 0$ up to zero (Figure 3B).

At the edges of the trap, the utilization rate is continuous at the lower threshold but discontinuous at the upper. Continuity is guaranteed by the monetary policy in place. Under the passive monetary policy rule, the relationship between the policy and the natural rate can be expressed in terms of the utility returns as $-\mu_\Lambda = \max\{-\mu_{\bar{\Lambda}}, 0\}$. Therefore, in the absence of discontinuities on the real side, a continuous natural rate, capacity utilization rate and policy rate at the ZLB are mutually consistent in equilibrium. Discontinuity is another consequence of the kink in the leverage multiple (see Subsection 4.2 for details). The kink makes the natural rate jump at the upper threshold from negative to positive terrain. In turn, the jump makes the policy rate jump as well but from zero to the positive rate. The utilization rate then must also jump in equilibrium: Otherwise, capacity utilization would not be consistent with the sudden (i.e., discontinuous) fluctuations around the threshold of the interest rates.[15]

Relative to the economy without the constraint, intermediary profitability falls throughout the trap. This is a joint consequence of capacity underutilization and the artificially high policy rate. The two elements combined create a double-whammy effect that compresses intermediary excess returns. Capacity underutilization depresses the rate of return on physical capital, while the high policy rate increases deposit funding costs.

Systemic risk increases throughout the trap as well. This is mainly because of the larger cliff at the upper threshold in the capital return of the marginal investor. Specifically, from the lower to right below the upper threshold, the return falls to $a_h u < a_h$ while right above the upper threshold and beyond, the return remains at 1. The more pronounced cliff renders the price of physical capital more sensitive to changes in the wealth share, especially around the upper threshold, where the utilization rate jumps by more. The more pronounced cliff then renders the price more sensitive to shock dZ as well. The larger sensitivity to the shock fuels amplification effects on financial conditions (Figure 3C) and, ultimately, increases systemic risk, $\sigma_\eta = (\phi - 1)(\sigma_q + \sigma)$. Formally, price-state elasticity ε_q increases around the threshold, which boosts both shock amplification factor σ_q/σ, with

$$\frac{\sigma_q}{\sigma} = \frac{(\phi - 1)\,\varepsilon_q}{1 - (\phi - 1)\,\varepsilon_q}\,, \tag{29}$$

and systemic risk factor, $\sigma_\eta/\sigma = (\phi - 1)(1 + \sigma_q/\sigma)$.[16]

The overall cycle is also more unstable (Figure 3D). Notably, the frequency and severity of busts increase while the frequency of booms falls. This is a joint consequence of the fall in intermediary profitability and the increase in systemic risk. Also, the exit from and entry to booms are markedly more turbulent. Upon exit, for instance, economic activity sinks abruptly (i.e., discontinuously), not because disruptions in financial intermediation destroy production capacity, but because aggregate demand falls drastically (i.e., again, discontinuously) below the remanent capacity after destruction. The aggregate demand setback renders aggregate output inversely U-shaped related to the wealth share during the trap. (Recall that $c/k + \iota = u\zeta$.) The output level and output risk nonetheless is more negatively correlated throughout the cycle.

Regarding the trend, its growth rate also falls throughout. This is because capacity underutilization deflates the dividend return on physical capital on impact—the price of physical capital falls throughout because it is a present discounted value of the returns. The depressed

[15]Note that the jump in the utilization rate is feasible because production decisions are fully flexible.

[16]The amplification factor, i.e. σ_q/σ, follows from combining risk quantities $\sigma_q = \varepsilon_q \sigma_\eta$ and $\sigma_\eta = (\phi - 1)(\sigma_q + \sigma)$. The factor measures the degree of shock amplification in $dq/q = \mu_q dt + \sigma_q dZ$. See footnote 17 for an alternative, more intuitive derivation of the factor. Note that in the frictionless economy, $\sigma_q/\sigma = 0$ is null.

growth trend, together with the exacerbated disruptions in financial markets, shape an interaction between liquidity traps, stagnation in economic activity, and financial instability that is consistent with Summers (2014b).

Lastly, the natural rate falls throughout as well, but it does so proportionally more during the trap. The three channels in the rate's decomposition contribute to the decline. The fall in the expected growth trend depresses the natural rate directly. The fall in intermediary profitability does so indirectly via a lower expected recovery pace of net production capacity after reinvestment (i.e., $\mu_{\zeta-\iota}$). The increase in systemic risk does so indirectly as well, but via a higher counterfactual (detrended) consumption risk, $\sigma_{\zeta-\iota} = \varepsilon_{\zeta-\iota}\sigma_\eta$.

All in all, as far as financial markets are concerned, liquidity traps worsen intermediary profitability, increase systemic risk, and exacerbate instability in the markets. The risk increase further depresses the natural rate, which in turn exacerbates the aggregate demand setback and the severity of traps. Relative to the economy without the ZLB, systemic risk reaches higher peaks, the natural rate reaches lower troughs, and the interactions between the two are stronger.

5.3 The Financially Regulated Economy

Macro-prudential policy can improve financial stability over the unregulated economy. It can also improve macroeconomic stabilization and social welfare even if its single objective is safeguarding financial stability. These improvements are possible because of pecuniary externalities in financial intermediation and an aggregate demand externality. I elaborate on the externalities, formalize the policy objectives, and conduct the policy analysis in what follows.

5.3.1 Externalities

The externalities can be regarded as the many channels through which financial intermediaries collectively affect each others' payoffs via asset prices or rates of return with their individual leverage decisions. The pecuniary externalities exist because of the financial frictions. These externalities are similar to those in Van der Ghote (2020). They are also three in total. The aggregate demand externality exists because of the ZLB constraint on nominal rates.

A first pecuniary externality operates through the response of the price of physical capital to shock dZ. This externality exists because financial intermediaries do not internalize the collective effect of their individual leverage decisions on the capital gain/loss rate, dq/q; the capital rate of return, dR_f; the rate of change of each other's net worth, dn_f/n_f; the rate of change of the wealth share, $d\eta/\eta$; and the infinite feedback loop between those rates that ensues. The externality can be measured by shock amplification factor σ_q/σ. It can be interpreted as the standard fire-sale externality in the seminal works of Gromb and Vayanos (2002) and Lorenzoni (2008).[17]

[17] In more detail, the shock triggers stochastic changes in dk/k by σdZ, which in turn trigger stochastic changes in dR_f by the same amount, which combined trigger stochastic changes in dn_f/n_f and $d\eta/\eta$ by $\phi\sigma dZ$ and $(\phi-1)\,\sigma dZ$, respectively. The change in $d\eta/\eta$ triggers stochastic changes in dq/q by $(\phi-1)\,\varepsilon_q\sigma dZ$, which in turn

The policy-based limit can reduce the externality. By doing so, mechanically, the limit mitigates shock amplification and hence keeps the economy more stable around the stochastic steady state. Pushing the intervention to the limit, forbidding leverage eliminates the externality, the amplification factor, systemic risk, and the endogenous risk quantities. This keeps the economy stable at the state. Formally, if $\Phi = 1$ always, then $\sigma_q/\sigma = 0$, $\sigma_\eta = 0$, and $\sigma_x = \varepsilon_x \sigma_\eta = 0$ always as well, which ensures that $\lim_{\eta \to \eta_{ss}^-} G(\eta) = 0 < 1 = G(\eta_{ss})$.

A second pecuniary externality instead operates through the level of the price of physical capital. During booms, financial intermediaries could collectively reduce their leverage to render households the marginal investors. This would depress the price of physical capital, not only on impact, but also throughout. A lower price would boost gross dividend yields $1/q$ and it would also reduce the expected rate of return on investment, $I'(\iota)\, q$. Regardless of the final effect on profitability, individual intermediaries do not internalize the collective effect of their leverage on the price either. The policy-based limit can intensify this externality by extending the phases in which the intermediaries are financially constrained and households are the marginal investors. In stagnant economies with sufficiently little investment—such as the ones considered in the paper—the positive effect dominates, which ensures that the lower price would exert upward pressures on collective profitability. In a related framework, Di Tella (2019) finds a similar pecuniary externality that also operates through the level of the price of the risky asset.

The third, and last, pecuniary externality operates through the market-based limit. This externality exists because of the specifics of the limited enforcement problems in deposits markets and IC portfolio constraint. In particular, an improvement in profitability boosts the Tobin's Q, increases trustworthy and thus allows the intermediaries to take on more leverage when the market-based limit is binding. Neither do the intermediaries internalize the collective effect of their leverage on each others' Tobin's Q nonetheless. A policy-based limit that restricts leverage appropriately around the upper threshold intensifies this externality. I will provide an example of such a policy below when I study the optimal policy.

Lastly, the aggregate demand externality exists only if the ZLB constraint is occasionally binding. In such economies, the intermediaries do not internalize the collective effect of their leverage on the double whammy either. The policy-based limit can reduce this externality by containing systemic risk and boosting the natural rate. Notably, in an economy without investment and a positive efficient natural rate, forbidding leverage eliminates the externality. This is because the intervention keeps the natural rate stable at efficiency by eliminating systemic risk and the fluctuations in detrended consumption.

Besides the externalities, the policy-based limit also affects production capacity. It does so, in particular, by shrinking the capacity on impact below the level that is consistent with $\phi = \min\{\lambda v, 1/\eta\}$. The closer the level to efficiency, the lower the welfare losses from the

trigger an infinite feedback loop between dq/q, dR_f, dn_f/n_f, and $d\eta/\eta$. The change in dq/q in each round of the loop is $[(\phi - 1)\,\varepsilon_q]^m\, \sigma dZ$, with $m \in \mathbb{N}$. The sum of the changes in dq/q over the rounds of the loop yields the shock amplification factor (see formula (29)). The externality exists because financial intermediaries take $dq/q = \mu_q dt + \sigma_q dZ$ as given.

reductions in detrended consumption. This is because consumption has decreasing marginal utility returns.

5.3.2 Policy Objectives

The extent to which macro-prudential policy is concerned with the externalities and the on-impact distortions in production capacity depends on the policy's objective. In what follows, I consider an economy without investment and restrict attention to logarithmic preferences. That is, I set $I_0 = 0$ and $\gamma = 1$. This specification allows me to additively decompose social welfare in three terms as follows: $W = W_M + W_F + \rho^{-1} \ln k$, with the values on the RHS satisfying Hamilton-Jacobi-Bellman (HJB) equations:

$$\rho W_M = \ln u + \frac{\partial W_M}{\partial \eta} \mu_\eta \eta + \frac{1}{2} \frac{\partial^2 W_M}{(\partial \eta)^2} (\sigma_\eta \eta)^2 \ , \tag{30}$$

and

$$\rho W_F = \ln \zeta + \frac{\partial W_F}{\partial \eta} \mu_\eta \eta + \frac{1}{2} \frac{\partial^2 W_F}{(\partial \eta)^2} (\sigma_\eta \eta)^2 \ , \tag{31}$$

respectively.[18]

Mapping W_M is the present discounted value of the utility flows from capacity utilization. Mapping W_F is the corresponding utility value of the flows from production capacity. The sum of the two is the utility value of the flows from detrended consumption. The three values are conditional on the economy's being at state η. Social welfare in the long run is $\int W(\eta)\, dG(\eta)$. I define macroeconomic stabilization as $\int W_M(\eta)\, dG(\eta)$ and financial stability as $\int W_F(\eta)\, dG(\eta)$. These definitions are consistent with the usual objectives of monetary and macro-prudential policy, respectively, in actual economies. The objectives are also welfare-based in the setup. Notably, financial stability and social welfare coincide in the economy without the ZLB. In the subsection below, I consider a macro-prudential policy that is concerned with either financial stability or social welfare. I examine the effects of the optimal policy on systemic risk, the natural rate, and the equilibrium under the two objectives. Also, I contrast the effects between the objectives. After the policy analysis, I clarify which results require the decomposition and which do not, and thus also hold under general parameter values.

5.3.3 Optimal Interventions

I derive the optimal policies numerically. To do so, I abstract away from policy-based limits that create discontinuities in the leverage multiple. Also, I restrict attention to limits that are given by $\Phi = \psi \min\{\lambda v, 1/\eta\}$ when they are binding, with $\psi < 1$ being a polynomial mapping of the

[18] I derive the HJB equations in the Online Appendix. To do so, first, I derive the HJB equation for social welfare under general parameter values. Mappings W, W_M, and W_F are continuous in the wealth share. This is because the mappings are present discounted values. In general, the mappings have a kink at $\eta = \bar{\eta}$. The kink follows from the kink in the leverage multiple and the jump in the utilization rate. Under the invariant distribution, the mappings are twice continuously differentiable almost surely.

wealth share. This functional form can be interpreted as a percentage limit over the natural quantity of leverage. The limit can vary over the cycle but its variation is limited by a parametric function of financial conditions. The two restrictions combined reduce the dimensionality of the macro-prudential optimization problem from an infinite to a finite number. Thus they render the problem tractable. I postulate the same properties on the Tobin's Q as in Subsection 4.2.

Financial Stability A macro-prudential policy that is concerned only with financial stability restricts leverage occasionally over the cycle. In particular, it does so only during phases in which the market-based limit would otherwise have been occasionally binding. Formally, $\psi < 1$ is active only when $\eta \in (\eta_L, \eta_H) \supset \bar{\eta}$, with policy thresholds $0 < \eta_L < \eta_H < 1$ satisfying that $\psi(\eta_L) = \psi(\eta_H) = 1$ (Figure 4A). The closer the wealth share to state $\bar{\eta}$, the tighter the percentage limit. The optimal policy behaves similarly over the cycle regardless of the binding status or degree of tightness of the ZLB constraint in equilibrium. The effects of the policy on the externalities and the natural rate nonetheless depend on those two properties. I elaborate on these results in what follows by analyzing the following three, exhaustive, mutually exclusive cases. The cases arise depending on the parameter values.

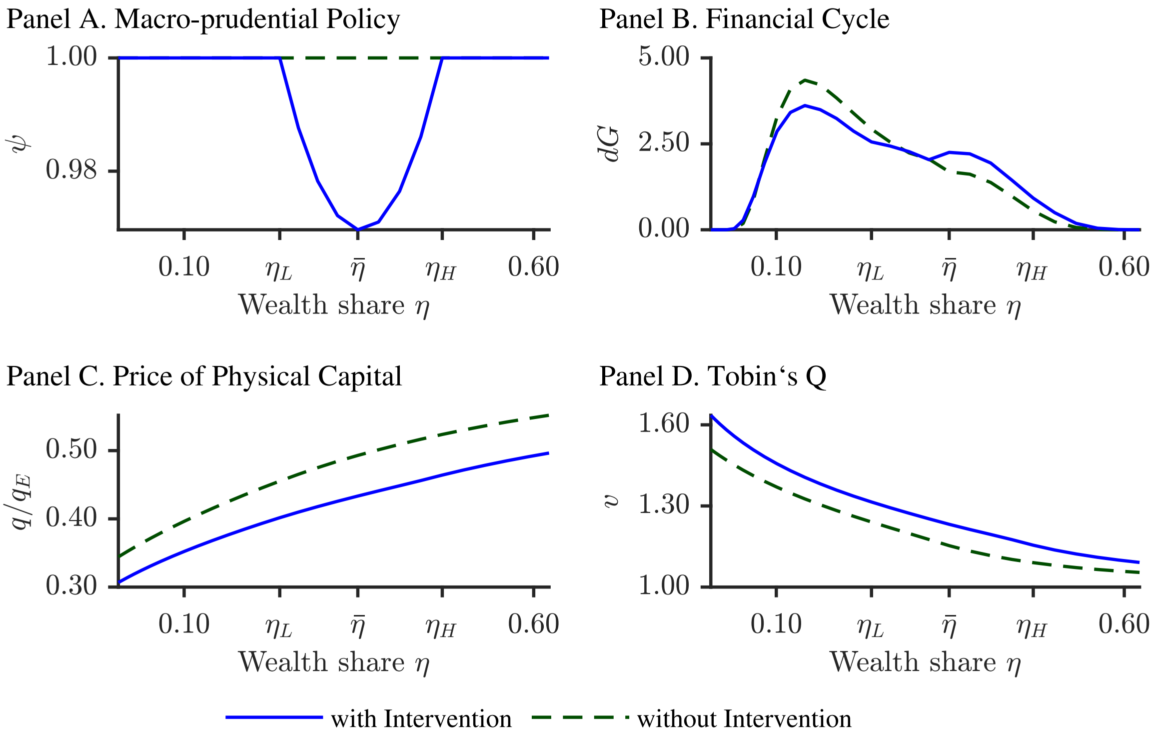
In a first case, the values are such that the natural rate under laissez faire lies above the ZLB throughout. Under laissez faire, therefore, the ZLB constraint is irrelevant and the aggregate demand externality does not exist. I postulate that the constraint is also irrelevant under the optimal policy. The optimal policy is then the same as in the economy without the ZLB. In particular, the optimal limit reduces risk-taking on impact and thus mitigates shock amplification. The reduced amplification helps keep the economy more stable around better capitalized phases with higher wealth shares (Figure 4B).[19] Also, the optimal limit depresses the price of physical capital throughout (Figure 4C). The lower price improves intermediary profitability; speeds the aggregate recapitalization of the intermediaries, and boosts the Tobin's Q throughout as well (Figure 4D). The speed-up in recapitalization further helps keep the economy more stable around the better-capitalized phases. The higher Tobin's Q increases leverage and stimulates production capacity, precisely when the intermediaries as a whole are poorly capitalized and the market-based limit is binding (i.e., $\eta \leq \eta_L$). In such poorly capitalized phases detrended consumption thus falls by less, which further helps smooth consumption over the cycle. Overall, the reduction in systemic risk and improvement in financial stability exert upward pressures on the natural rate. The upward pressures reinforce the slackness of the ZLB constraint and thus confirm the ZLB irrelevance.[20]

In a second case, the parameter values are such that (i) the natural rate under laissez faire

[19]The blue lines in Figure 4 plot the equilibrium under the optimal policy in an economy with an irrelevant (i.e., always slack) ZLB constraint. See the notes in the figure for details.

[20]Below $\eta = \eta_L$, the optimal limit is slack because natural production capacity is already way below efficiency when the intermediaries collectively are poorly capitalized. Above $\eta = \eta_H$, the limit is also slack but because shock amplification is already reasonably low when the intermediaries collectively are richly capitalized. Recall that restricting leverage below $\phi = \min\{\lambda v, 1/\eta\}$ reduces production capacity on impact as well as aggregate consumption.

Figure 4: Equilibrium with Macro-prudential Policy in Case 2

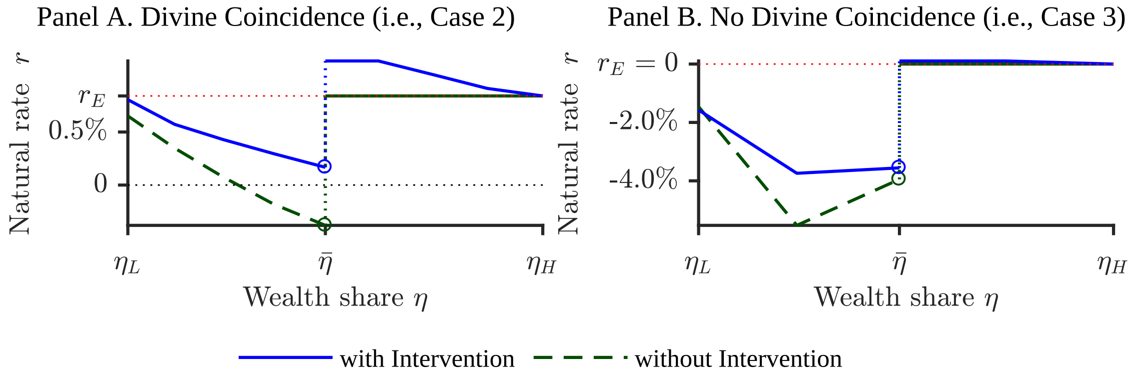


Notes: The figure considers the case in which complementarity is sufficiently strong to generate the coincidence (i.e., Case 2). The parameter values are the same as in the baseline (Table 1) except for $I_0 = 0$, $\gamma = 1$, and $\rho = 1.2\%$. The latter value is set to ensure that the coincidence exists. All of the other parameter values fixed, if $\rho \ll 1.2\%$ is sufficiently low, complementarity exists but the coincidence does not (i.e., Case 3). If $\rho \gg 1.2\%$ is sufficiently high, neither does complementarity nor the coincidence exist (i.e., Case 1). In cases 1 and 2, the optimal policy is the same as in the counterpart economy without the ZLB constraint. The equilibrium outcome under the optimal policy is also the same. In case 3, the optimal policy in general is tighter (see Table 3 for details).

falls below the ZLB at least occasionally over the cycle, but (ii) the same rate under the optimal policy instead lies above throughout. In this other case, in principle, the ZLB constraint and the aggregate demand externality matter for the design of the optimal policy. However, actually, they do not. This is because the optimal regulation of the pecuniary externalities under a postulate of a nonexistent aggregate demand externality is consistent with the postulate in equilibrium. Put simply, complementarity between the two types of externalities is sufficiently strong to generate a divine coincidence between the two. The intervention and the equilibrium outcome are then also the same as in the counterpart economy without the constraint (Figure 4 and Figure 5A).

In the third, and last, case, the parameter values are such that the natural rate does not satisfy the conditions of the previous two cases. The ZLB constraint and the aggregate demand externality then matter for the optimal policy design. Under this objective, however, they do so only to the extent to which financial stability is concerned. The optimal policy is tighter than in the economy without the constraint (Table 3). This is because of the double-whammy effect

Figure 5: Effect of Macro-prudential Policy on Natural Rate



Notes: Panel A uses the same parameter values as Figure 4. Panel B uses those same values but with $\rho = \sigma^2 = 0.36\% < 1.2\%$. Value $\rho = 0.36\%$ is such that in the economy without investment and with logarithmic preferences for consumption (i.e., $I_0 = 0$ and $\gamma = 1$), the efficient natural rate, $r_E = \rho - \sigma^2 = 0$, is null. Panel B considers the case in which complementarity exists, but it is not sufficiently strong to generate the coincidence.

that compresses excess returns during the traps. Under that benchmark intervention, the double whammy is left unregulated, because the intervention internalizes neither the aggregate demand externality nor the liquidity trap. Starting from the benchmark, a tightening of leverage reduces shock amplification and systemic risk. These reductions exert upward pressures on the natural rate and thus relax the ZLB constraint. A softer ZLB renders deposits rates not so artificially high, reduces capacity underutilization, and, ultimately, weakens the double whammy effect. On the margin, these improvements are crucial as well for enhancing financial stability. Unlike in the first case, but like in the second, complementarity between the two types of externality exists. Unlike the second case, the divine coincidence does not.

Social Welfare The same three cases hold if macro-prudential policy is instead concerned with social welfare. The main results are indeed similar. Under this other objective, however, the results can rather be stated in terms of a potential complementarity between financial stability and macroeconomic stabilization. In the first case, the complementarity does not exist. This is because monetary policy can guarantee macroeconomic stabilization on its own. Macro-prudential policy limits itself to safeguarding financial stability. This is the typical behavior of policy in environments with “normal” levels of interest rates. In the other two cases, complementarity instead exists. In the second case, the complementarity is sufficiently strong to generate a divine coincidence between the two policy objectives. The natural rate remains above the ZLB throughout the cycle, but only because of the improvement of macro-prudential policy on financial stability. Macro-prudential policy also limits itself to safeguarding financial stability. Its benefits on macroeconomic stabilization are unintended, and they are also latent, in the sense that liquidity traps do not occur in equilibrium. Lastly, in the third case, the complementarity is also present, but it is not strong enough to generate the coincidence. Macro-prudential policy further restricts leverage relative to its counterpart with the financial stability objective (Table

3). This is because the counterpart policy does not internalize the direct losses in detrended consumption from capacity underutilization.

Table 3 — Macro-prudential Intervention in Case 3

	Limit on leverage		Gains over laissez faire in...		
	$\Pr[\psi<1]$	$E[\psi\vert\psi<1]$	$E[W_F]$	$E[W_M]$	$E[W]$
Panel A. Economy without ZLB	7.8%	99.0%	0.57%	0.00%	0.57%
Panel B. Economy with ZLB					
Maximize financial stability	8.6%	98.3%	0.73%	0.27%	1.00%
Maximize social welfare	9.8%	97.5%	0.66%	0.36%	1.02%

Notes: The parameter values are the same as in Figure 5, Panel B. Namely, the values are the same as those in Table 1, but with $I_0 = 0$, $\gamma = 1$, and $\rho = \sigma^2 = 0.36\% < 1.2\%$. The table considers the case in which complementarity exists but the coincidence does not (i.e., Case 3). Gains over laissez faire are measured in terms of permanent, percentage increases in annual consumption (a.k.a., annual consumption equivalent).

5.3.4 Discussion

None of the properties that define the three cases requires the welfare decomposition. Neither do the optimal policy in the first two cases nor the optimal policy in the third case under the welfare objective. The optimal policy in the third case under the objective of financial stability is the single result that requires the decomposition. The complementarity and divine coincidence results do not require an unconstrained policy-based limit. However, in general, constraints on the limit reduce the parameter region in which complementarity or divine coincidence exist. This is because constrained optimal policies improve financial stability by less than their unconstrained counterpart. The upward pressures on the natural rate thus are lower as well.

6 Conclusion

In this paper, I reveal a novel mechanism through which macro-prudential policy helps sustain macroeconomic stabilization in low interest rate environments. The mechanism is the result of the inverse relationship between systemic risk and the natural rate. Also, I show that the resulting improvement on macroeconomic stabilization follows naturally, as a by-product of the improvement on financial stability. Lastly, I find that a divine coincidence between financial stability and macroeconomic stabilization exists, provided that the natural rate is secularly low, but not too low.

To conduct the policy analysis, I restrict attention to a macro-prudential instrument that takes the form of a state-contingent limit on leverage. The main results also hold nonetheless for the many other macro-prudential instruments that curb excessive risk-taking. Notable examples include loan-to-value (LTV) and payment-to-income (PTI) ratios to households or firms. Assessing the relative contribution to macroeconomic stabilization of the many instruments remains for future research.

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Online Appendix

The online Appendix has two parts. The first part solves the portfolio optimization problems of financial intermediaries and households, in that order. The second part derives the ordinary differential equation system (ODEs) that analytically characterizes the solution to the model, the invariant density function, and the Hamilton-Jacobi-Bellman (HJB) equations that analytically characterize the policy objectives and social welfare. The derivations in the second part are done for both the baseline model (Section 2) and the model with the zero lower bound (ZLB) constraint (Section 5).

Portfolio Problems

Financial Intermediaries Value function V_t satisfies

$$e^{-\theta t}\Lambda_t V_t + \int_0^t \theta e^{-\theta s}\Lambda_s n_{f,s} ds = E_t \int_0^\infty \theta e^{-\theta s}\Lambda_s n_{f,s} ds \; . \tag{32}$$

The RHS is a conditional expectation of a random variable. Therefore, its drift process is null. From first, applying Ito's Lemma to the LHS and then, equating the resulting drift process to zero, it follows this HJB equation

$$\begin{aligned} 0 \; &= \; \max_{\iota_{f,t},\phi_t \geq 0} \left\{ \frac{\theta}{v_t} + \mu_{\Lambda,t} + \mu_{v,t} + \mu_{n_f,t} + \sigma_{\Lambda,t}\sigma_{v,t} + \sigma_{\Lambda,t}\sigma_{n_f,t} + \sigma_{v,t}\sigma_{n_f,t} - \theta \right\} , \\ & \quad\; \text{subject to} : \phi_t \leq \lambda v_t \; , \end{aligned} \tag{33}$$

with

$$\mu_{n_f,t} = \left[\frac{1-\iota_{f,t}}{q_t} + \mu_{q,t} + I\left(\iota_{f,t}\right) - \delta + \sigma_{q,t}\sigma - r_t \right] \phi_t + r_t \; , \tag{34}$$

$$\sigma_{n_f,t} = \left(\sigma_{q,t} + \sigma\right)\phi_t \; , \tag{35}$$

being the drift and diffusion processes of intermediary net worth $n_{f,t}$, respectively, and $\phi_t \equiv q_t k_{f,t}/n_{f,t}$ the leverage multiple. The rest of the objects in the HJB equation are the same as in the paper. To derive the equation, I use the conjecture made in the paper that $V_t \equiv v_t n_{f,t}$.

Financial intermediaries take all of the processes in the equation except $\iota_{f,t}$, ϕ_t, $\mu_{n_f,t}$, and $\sigma_{n_f,t}$ as given. The first order condition (FOC) with respect to $\iota_{f,t}$ is

$$I'\left(\iota_{f,t}\right) = \frac{1}{q_t} \; . \tag{36}$$

The FOC with respect to ϕ_t is

$$\phi_t = \begin{bmatrix} \lambda v_t & \text{if } \alpha_{f,t} > 0 \\ \beta_{f,t} & \text{if } \alpha_{f,t} = 0 \\ 0 & \text{if } \alpha_{f,t} < 0 \end{bmatrix} , \tag{37}$$

with $\beta_{f,t}$ being a real number in interval $[0, \lambda v_t]$, and

$$\alpha_{f,t} \equiv \frac{1 - \iota_{f,t}}{q_t} + \mu_{q,t} + I\left(\iota_{f,t}\right) - \delta + \sigma_{q,t}\sigma - r_t + \left(\sigma_{q,t} + \sigma\right)\left(\sigma_{\Lambda,t} + \sigma_{v,t}\right) , \tag{38}$$

as in the paper.

I set $r_t = -\mu_{\Lambda,t}$. I will derive this relationship below. This substitution is valid here because the intermediaries take the two processes as given. From substituting the FOCs into the HJB equation, it follows that

$$\alpha_{f,t}\phi_t + \mu_{v,t} + \sigma_{\Lambda,t}\sigma_{v,t} + \frac{\theta}{v_t} - \theta = 0 , \tag{39}$$

as in the paper.

Expressions (33) to (39) are the same for all of the intermediaries. Therefore, processes $\iota_{f,t}$, ϕ_t, and v_t are the same for all of the intermediaries as well. In equilibrium, a representative financial intermediary thus exists. Drift and diffusion processes $\mu_{n_f,t}$ and $\sigma_{n_f,t}$ do not depend on individual net worth. Hence, $V_t = E_t \int_t^\infty \theta e^{-\theta(s-t)} \left(\Lambda_s/\Lambda_t\right) n_{f,s} ds$ is linear in current individual net worth $n_{f,t}$. This property verifies the conjecture that $V_t \equiv v_t n_{f,t}$.

Households Value function W_t satisfies the usual HJB equation

$$0 = \max_{c_t, \iota_t, k_{h,t} \geq 0} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{1}{dt} E_t\left[dW_t\right] - \rho W_t \right\} . \tag{40}$$

I conjecture that the function also satisfies

$$W_t = W\left(n_{h,t}, J_t\right) , \tag{41}$$

with $W : \mathbb{R}^2 \to \mathbb{R}$ being a twice continuously differentiable function and $J_t \in \mathbb{R}$ a real-valued process that evolves over time stochastically according to $dJ_t/J_t = \mu_{J,t}dt + \sigma_{J,t}dZ_t$. Process J_t is endogenous in the model, but households take the process as given. Shock dZ_t is the same Brownian disturbance as that in the law of motion of physical capital.

From first, applying Ito's Lemma to both sides of equation (41), and then, substituting the resulting expression into (40), it follows this HJB equation

$$\rho W_t = \max_{c_t, \iota_t, k_{h,t} \geq 0} \left\{ \begin{array}{l} \frac{1}{1-\gamma} c_t^{1-\gamma} + \frac{\partial W_t}{\partial n_{h,t}} \mu_{n_h,t} n_{h,t} + \frac{\partial W_t}{\partial J_t} \mu_{J,t} J_t + \\ \frac{1}{2} \frac{\partial_t^2 W_t}{(\partial n_{h,t})^2} (\sigma_{n_h,t} n_{h,t})^2 + \frac{\partial^2 W_t}{\partial J_t \partial n_{h,t}} \sigma_{J,t} J_t \sigma_{n_h,t} n_{h,t} + \frac{1}{2} \frac{\partial^2 W_t}{(\partial J_t)^2} (\sigma_{J,t} J_t)^2 \end{array} \right\}, \tag{42}$$

with

$$\mu_{n_h,t} n_{h,t} = \left[\frac{1 - \iota_{h,t}}{q_t} + \mu_{q,t} + I(\iota_{h,t}) - \delta + \sigma_{q,t}\sigma - r_t \right] q_t k_{h,t} + r_t n_{h,t} + \tau_t - c_t , \tag{43}$$

$$\sigma_{n_h,t} n_{h,t} = (\sigma_{q,t} + \sigma) q_t k_{h,t} . \tag{44}$$

being the drift and diffusion processes of the net worth of households, respectively.

Households take all of the processes in the equation as given, except c_t, $\iota_{h,t}$, $k_{h,t}$, $\mu_{n_h,t}$, and $\sigma_{n_h,t}$. The FOCs are

$$c_t^{-\gamma} = \frac{\partial W_t}{\partial n_{h,t}} , \tag{45}$$

$$I'(\iota_{h,t}) = \frac{1}{q_t} , \tag{46}$$

$$0 = \left\{ \frac{\partial W_t}{\partial n_{h,t}} \left[\frac{1-\iota_{h,t}}{q_t} + \mu_{q,t} + I(\iota_{h,t}) - \delta + \sigma_{q,t}\sigma - r_t \right] + \left[\frac{\partial^2 W_t}{(\partial n_{h,t})^2} \sigma_{n_h,t} n_{h,t} + \frac{\partial^2 W_t}{\partial J_t \partial n_{h,t}} \sigma_{J,t} J_t \right] (\sigma_{q,t} + \sigma) \right\} q_t k_{h,t} . \tag{47}$$

The optimality condition in the paper for the investment rate follows directly from the second FOC. The other two optimality conditions follow from applying the same methodology as in Cox, Ingersoll and Ross (1985). Specifically, first evaluate the FOCs in (42); second, differentiate the resulting expression with respect to $n_{h,t}$; third, rearrange the resulting expression accordingly to obtain the inter-temporal condition between consumption and deposits, $-\mu_{\Lambda,t} = r_t$, and the optimal capital choice

$$q_t k_{h,t} = \left[\begin{array}{ll} +\infty & \text{if } \alpha_{h,t} > 0 \\ \beta_{h,t} & \text{if } \alpha_{h,t} = 0 \\ 0 & \text{if } \alpha_{h,t} < 0 \end{array} \right. , \tag{48}$$

with $\beta_{h,t} \in [0, +\infty)$, and

$$\alpha_{h,t} \equiv \frac{a_h - \iota_{h,t}}{q_t} + \mu_{q,t} + I(\iota_{h,t}) - \delta + \sigma_{q,t}\sigma - r_t + (\sigma_{q,t} + \sigma) \sigma_{\Lambda,t} . \tag{49}$$

Markov Equilibrium

In what follows, I omit subindex $_t$ when denoting variables. First, I derive the pertinent objects for the baseline model. Then, I do so for the model with the ZLB constraint. For the second model, first, I consider the economy under laissez faire, and then, I consider the economy with macro-prudential policy.

Baseline Model

ODEs The equilibrium pricing equation for physical capital is

$$\begin{bmatrix} \alpha_h \equiv \frac{a_h - \iota}{q} + \mu_q + I(\iota) - \delta + \sigma_q \sigma - r + (\sigma_q + \sigma)\,\sigma_\Lambda = 0 & \text{if } \eta < \bar{\eta} \\ \alpha_f \equiv \frac{1-\iota}{q} + \mu_q + I(\iota) - \delta + \sigma_q \sigma - r + (\sigma_q + \sigma)(\sigma_\Lambda + \sigma_v) = 0 & \text{if } \eta \geq \bar{\eta} \end{bmatrix}, \tag{50}$$

with $\zeta = a_h + (1 - a_h)\,\phi\eta$, $\phi = \min\{\lambda v, 1/\eta\}$, $\iota = I'^{-1}(q)$, $\Lambda = e^{-\rho t}\left[(\zeta - \iota)\,k\right]^{-\gamma}$, $r = -\mu_\Lambda$, and $\bar{\eta} \in (0,1)$ such that $\lambda v(\bar{\eta})\,\bar{\eta} = 1$. The equilibrium pricing equation for value v is

$$\alpha_f \phi + \frac{\theta}{v} + \mu_v - \gamma + \sigma_v \sigma_q = 0\,. \tag{51}$$

Ito's Lemma implies that

$$\mu_x = \varepsilon_x \mu_\eta + \frac{1}{2}\varepsilon_{x_\eta}\varepsilon_x \sigma_\eta^2\,, \tag{52}$$

$$\sigma_x = \varepsilon_x \sigma_\eta\,, \tag{53}$$

with $\varepsilon_x \equiv x_\eta \eta / x$ and $x_\eta \equiv \partial x / \partial \eta$ being the elasticity and the first-order derivative, respectively, of underlying variable x with respect to η.

Let $\xi \equiv (\zeta - \iota)^{-\gamma}$ and let $\tilde{q} \equiv \xi q$. Ito's Lemma implies that

$$\mu_\Lambda = -\rho + \mu_\xi - \gamma\left[I(\iota) - \delta\right] - \sigma_\xi \gamma \sigma + \frac{1}{2}\gamma(\gamma + 1)\,\sigma^2\,, \tag{54}$$

$$\sigma_\Lambda = \sigma_\xi - \gamma\sigma\,, \tag{55}$$

and that

$$\mu_q = \mu_{\tilde{q}} - \mu_\xi - \sigma_{\tilde{q}}\sigma_\xi\,. \tag{56}$$

The law of motion of the wealth share is

$$\mu_\eta = \frac{1-\iota}{q}\phi + \left[\mu_q + I(\iota) - \delta + \sigma_q\sigma - r - (\sigma_q + \sigma)^2\right](\phi - 1) - \left(\theta - \frac{\kappa}{\eta}\right), \tag{57}$$

$$\sigma_\eta = (\phi - 1)(\sigma_q + \sigma)\,. \tag{58}$$

I substitute $\sigma_q = \varepsilon_q \sigma_\eta$ into (58) to express σ_η as a function of the parameters, the wealth share, and functions of the wealth share. I obtain

$$\sigma_\eta = \frac{\phi - 1}{1 - (\phi - 1)\,\varepsilon_q}\sigma\,, \tag{59}$$

which allows me to also express σ_v, σ_ξ, and $\sigma_{\tilde{q}}$ as a function of the same objects.

I substitute (54) and (56) into (57). I obtain

$$\mu_\eta = \frac{1-\iota}{q}\phi - \left(\theta - \frac{\kappa}{\eta}\right) + \left[\mu_{\tilde{q}} - \sigma_{\tilde{q}}\sigma_\xi + (1-\gamma)\left[I(\iota)-\delta\right] + \sigma_q\sigma - \rho - \sigma_\xi\gamma\sigma + \frac{1}{2}\gamma(\gamma+1)\sigma^2 - (\sigma_q+\sigma)^2\right](\phi-1)\ . \tag{60}$$

I evaluate (52) at $x=\tilde{q}$. I substitute the resulting expression into (60) to obtain

$$\mu_\eta = \frac{1}{1-(\phi-1)\varepsilon_{\tilde{q}}} \times \left\{\frac{1-\iota}{q}\phi - \left(\theta - \frac{\kappa}{\eta}\right) + \left[\frac{1}{2}\varepsilon_{\tilde{q}_\eta}\varepsilon_{\tilde{q}}\sigma_\eta^2 - \sigma_{\tilde{q}}\sigma_\xi + (1-\gamma)\left[I(\iota)-\delta\right] + \sigma_q\sigma - \rho - \sigma_\xi\gamma\sigma + \frac{1}{2}\gamma(\gamma+1)\sigma^2 - (\sigma_q+\sigma)^2\right](\phi-1)\right\}\ . \tag{61}$$

This allows me to also express μ_η, μ_q, and μ_v as a function of the parameters, the wealth share, and functions of the wealth share.

Expressions (50), (51), (52), (53), (54), (55), (59), and (61) jointly determine an implicit second-order ODEs for $\{q, v\}$ in η. I impose the following boundary conditions

$$\lim_{\eta\to 1}\sigma_q = 0\ ,\quad \lim_{\eta\to 1}\frac{\partial\sigma_q}{\partial\eta} = 0\ ,\quad \lim_{\eta\to 1}\sigma_v = 0\ ,\quad \lim_{\eta\to 1}\frac{\partial\sigma_v}{\partial\eta} = 0\ . \tag{62}$$

These conditions ensure that endogenous risk quantities σ_q and σ_v vanish smoothly as the aggregate net worth of financial intermediaries approaches total wealth. More generally, the conditions impose that any endogenous risk quantity, $\sigma_x = \varepsilon_x\sigma_\eta$, behaves in such a manner. In a similar economy in which financial intermediaries face the same portfolio problem as in this model except for the specifics of the portfolio constraint, Maggiori (2017) imposes the same conditions.

Invariant Distribution Invariant density function dG solves Kolmogorov-Chapman forward equation

$$-\frac{\partial}{\partial\eta}\left[\mu_\eta\eta dG\right] + \frac{\partial^2}{\partial\eta^2}\left[(\sigma_\eta\eta)^2 dG\right] = 0\ . \tag{63}$$

Thus, dG satisfies

$$dG(\eta) \propto \frac{1}{(\sigma_\eta\eta)^2}\exp\left\{2\int_0^\eta \frac{\mu_{\tilde{\eta}}\tilde{\eta}}{(\sigma_{\tilde{\eta}}\tilde{\eta})^2}d\tilde{\eta}\right\} \text{ with } \int_0^1 dG(\eta)\,d\eta = 1\ . \tag{64}$$

HJB Equation Social welfare W satisfies HJB equation

$$\begin{aligned}\rho W &= \frac{1}{1-\gamma}\left[(\zeta-\iota)k\right]^{1-\gamma}+\frac{\partial W}{\partial\eta}\mu_\eta\eta+\frac{\partial W}{\partial k}\left[I(\iota)-\delta\right]k+ \\ &\quad \frac{1}{2}\frac{\partial W^2}{(\partial\eta)^2}(\sigma_\eta\eta)^2+\frac{\partial W^2}{\partial\eta\partial k}\sigma_\eta\eta\sigma k+\frac{1}{2}\frac{\partial W^2}{(\partial k)^2}(\sigma k)^2 \; .\end{aligned} \tag{65}$$

I conjecture that social welfare also satisfies $W=\tilde{W}k^{1-\gamma}$, with $\tilde{W}$ being a function of wealth share η alone. Function $\tilde{W}$ is the present discounted value of utility flows from detrended consumption. Under the conjecture, the HJB equation reduces to

$$\rho\tilde{W}=\frac{(\zeta-\iota)^{1-\gamma}}{1-\gamma}+\frac{\partial\tilde{W}}{\partial\eta}\left[\mu_\eta\eta+(1-\gamma)\sigma_\eta\eta\sigma\right]+(1-\gamma)\left[I(\iota)-\delta-\frac{\gamma}{2}\sigma^2\right]\tilde{W}+\frac{1}{2}\frac{\partial^2\tilde{W}}{(\partial\eta)^2}(\sigma_\eta\eta)^2 \; . \tag{66}$$

Expressions (66), (52), (53), (54), (55), (59), and (61) jointly determine an implicit second-order ODEs for $\tilde{W}$ in η. I impose the following boundary conditions

$$\lim_{\eta\to 1}\sigma_{\tilde{W}}=0\;,\quad \lim_{\eta\to 1}\frac{\partial\sigma_{\tilde{W}}}{\partial\eta}=0\;. \tag{67}$$

These conditions are consistent with (62). If $I(\iota)=\iota=0$ and $\gamma=1$, the HJB equation further reduces to

$$\rho\tilde{W}=\ln\zeta+\frac{\partial\tilde{W}}{\partial\eta}\mu_\eta\eta+\frac{1}{2}\frac{\partial^2\tilde{W}}{(\partial\eta)^2}(\sigma_\eta\eta)^2\;. \tag{68}$$

Model with ZLB Constraint

Economy without Macro-prudential Policy

I consider economies in which liquidity traps occur only when $\eta\in\left(\underline{\eta},\bar{\eta}\right)$, with $0<\underline{\eta}<\bar{\eta}<1$.

Outside the trap, the ODEs is the same as in the baseline model. Inside the trap, the equilibrium pricing condition for physical capital is

$$\left[\begin{array}{ll}\alpha_h\equiv\frac{ua_h-\iota}{q}+\mu_q+I(\iota)-\delta+\sigma_q\sigma+(\sigma_q+\sigma)\sigma_\Lambda=0 & \text{if }\eta\in\left(\underline{\eta},\bar{\eta}\right)\\ \alpha_f\equiv\frac{u-\iota}{q}+\mu_q+I(\iota)-\delta+\sigma_q\sigma+(\sigma_q+\sigma)(\sigma_\Lambda+\sigma_v)=0 & \text{if }\eta\in\left(\underline{\eta},\bar{\eta}\right)\end{array}\right. , \tag{69}$$

with $\Lambda=e^{-\rho t}\left[(u\zeta-\iota)k\right]^{-\gamma}$, $\bar{\Lambda}=e^{-\rho t}\left[(\zeta-\iota)k\right]^{-\gamma}$, $i=-\mu_\Lambda=0$, and $r=-\mu_{\bar{\Lambda}}<0$. I impose that the rate is continuous at $\eta=\underline{\eta}$ but may jump at $\eta=\bar{\eta}$. That is, $\lim_{\eta\to\underline{\eta}^+}u(\eta)=u\left(\underline{\eta}\right)=1$ and $\lim_{\eta\to\bar{\eta}^-}u(\eta)\le u(\bar{\eta})=1$. The rest of the variables and conditions in the ODEs are the same as in the baseline model except for the condition for μ_η. In particular, the drift process satifies

$$\mu_\eta=\frac{u-\iota}{q}\phi+\left[\mu_q+I(\iota)-\delta+\sigma_q\sigma-(\sigma_q+\sigma)^2\right](\phi-1)-\left(\theta-\frac{\kappa}{\eta}\right)\;. \tag{70}$$

Hence

$$\mu_\eta = \frac{1}{1-(\phi-1)\varepsilon_q}\left\{\frac{u-\iota}{q}\phi - \left(\theta - \frac{\kappa}{\eta}\right) + \left[\frac{1}{2}\varepsilon_{q_\eta}\varepsilon_q\sigma_\eta^2 + I(\iota) - \delta + \sigma_q\sigma - (\sigma_q+\sigma)^2\right](\phi-1)\right\}. \tag{71}$$

The condition for the invariant distribution is the same as in the baseline model. The HJB equation for detrended welfare is

$$\rho\tilde{W} = \frac{(u\zeta-\iota)^{1-\gamma}}{1-\gamma} + \frac{\partial\tilde{W}}{\partial\eta}\left[\mu_\eta\eta + (1-\gamma)\sigma_\eta\eta\sigma\right] + (1-\gamma)\left[I(\iota) - \delta - \frac{\gamma}{2}\sigma^2\right]\tilde{W} + \frac{1}{2}\frac{\partial^2\tilde{W}}{(\partial\eta)^2}(\sigma_\eta\eta)^2. \tag{72}$$

If $I(\iota) = \iota = 0$ and $\gamma = 1$, then

$$\rho\tilde{W} = \ln(u\zeta) + \frac{\partial\tilde{W}}{\partial\eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2\tilde{W}}{(\partial\eta)^2}(\sigma_\eta\eta)^2. \tag{73}$$

Thus, $\tilde{W} = \tilde{W}_M + \tilde{W}_F$, with

$$\rho\tilde{W}_M = \ln u + \frac{\partial\tilde{W}_M}{\partial\eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2\tilde{W}_M}{(\partial\eta)^2}(\sigma_\eta\eta)^2, \tag{74}$$

$$\rho\tilde{W}_F = \ln\zeta + \frac{\partial\tilde{W}_F}{\partial\eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2\tilde{W}_F}{(\partial\eta)^2}(\sigma_\eta\eta)^2. \tag{75}$$

Expressions (74), (52), (53), (54), (55), (59), and (61) jointly determine an implicit second-order ODEs for $\tilde{W}_M$ in η. The same expressions but with (75) instead of (74) do so for $\tilde{W}_F$. For both ODEs, I impose the following boundary conditions

$$\lim_{\eta\to1}\sigma_{\tilde{W}_j} = 0\,, \quad \lim_{\eta\to1}\frac{\partial\sigma_{\tilde{W}_j}}{\partial\eta} = 0\,, \text{ with } j\in\{M,F\}. \tag{76}$$

Economy with Macro-prudential Policy

I restrict attention to macro-prudential limits that are binding only when $\eta \in (\eta_L, \eta_H)$, with $0 < \eta_L < \bar{\eta} < \eta_H < 1$. I only consider limits that satisfy $\Phi = \psi\min\{\lambda v, 1/\eta\}$ when they are binding, with $\psi < 1$ being a polynomial mapping of η.

The equilibrium pricing condition for physical capital is

$$\left[\begin{array}{ll}\alpha_h \equiv \frac{a_h-\iota}{q} + \mu_q + I(\iota) - \delta + \sigma_q\sigma - r + (\sigma_q+\sigma)\sigma_\Lambda = 0 & \text{if } \eta < \eta_H \\ \alpha_f \equiv \frac{1-\iota}{q} + \mu_q + I(\iota) - \delta + \sigma_q\sigma - r + (\sigma_q+\sigma)(\sigma_\Lambda+\sigma_v) = 0 & \text{if } \eta \geq \eta_H\end{array}\right.. \tag{77}$$

The rest of the derivation is the same as in the economy without the policy, but with $\phi = \min\{\lambda v, \Phi, 1/\eta\}$.

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Alejandro Van der Ghote
European Central Bank, Frankfurt am Main, Germany; email: alejandro.van_der_ghote@ecb.europa.eu

Postal address 60640 Frankfurt am Main, Germany
Telephone +49 69 1344 0
Website www.ecb.europa.eu

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