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Bank contagion in general equilibrium

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Abstract

In this paper, I incorporate a complex network model into a state of the art stochastic general equilibrium framework with an active interbank market. Banks exchange funds one another generating a complex web of interbanking relations. With the tools of network analysis it is possible to study how contagion spreads between banks and what is the probability and size of a cascade (a sequence of defaults) generated by a single initial episode. Those variables are a key component to understand systemic risk and to assess the stability of the banking system. In extreme scenarios, the system may experience a phase transition when the consequences of one single initial shock affect the entire population. I show that the size and probability of a cascade evolve along the business cycle and how they respond to exogenous shocks. Financial shocks have a larger impact on contagion probability than real shocks that, however, are long lasting. Additionally I find that monetary policy faces a trade off between financial stability and macroeconomic stabilization. Government spending shocks, on the contrary, have smaller effects on both.

JEL Classification: E44, E32, E52, E58, D85.

Keywords: Contagion, Network Analysis, DSGE, Interbank Market, Heterogenous Agents.

Non-technical summary

The global financial crisis has shown how the web of mutual claims and obligations that ties together financial institutions is a key element to understand the propagation of financial shocks and sequences of defaults. Understanding the relationship between the structure of the financial market and the transmission of shocks has also relevant policy implications. For example, understanding the connections between agents is crucial to design a “resilient” system (i.e. a system able to absorb the default of several agents without systemic consequences) and to evaluate different tools to stop contagion. Understanding how financial agents are connected is also relevant to assess how exogenous shocks, including monetary policy shocks, increase or decrease the risk and size of contagion.
In the recent years, a growing literature has investigated those topics using different methodological approaches. One strand of research, for example, has focused on the so called “asset communality” explanation for systemic crises. This theory suggests that contagion arises because financial institutions invest in assets with highly correlated returns; when a negative shock occurs, they are all affected in a similar way. An alternative approach to explain contagion relies on modelling the web of financial connections between banks. The underlying assumption is that agents transmit contagion by direct links (i.e. if bank “a” defaults all banks that have claims against “a” sustain losses). The structure of those connections can explain how losses are transmitted from one financial institution to another and how shocks may be amplified moving through the financial system.
This paper provides three contributions to this last strand of literature. First, and most importantly, it augments a standard dynamic stochastic general equilibrium model (DSGE) with an endogenous network between banks. The network presented in this model evolves along the business cycle, reacting to exogenous shocks and endogenous macroeconomic developments. With the tools of network analysis it is possible to compute the likelihood and size of contagion at each point in time, which depend on the entire network structure. This is an important extension of standard DSGE models (the main workhorse models used by central banks for analysis and forecasting). Equipped with this additional tool, the model can be used to investigate the financial stability consequences of macroeconomic developments. Notably, the modelling technique presented here can be extended to other sectors of the economy where diffusion is important (for example technology or information).
Second, the paper exploits the general equilibrium dimension of the model to assess how different exogenous shocks affect the probability and size of contagion between banks. Most of the literature on contagion has so far focused on partial equilibrium models (i.e. models that focus on the financial

sector taking other sectors of the economy as given). This paper, on the contrary, is the first to assess the effects of aggregate shocks on the likelihood of financial contagion through the lens of a full macroeconomic model. The analysis is important to assess the impact of developments of real variables and monetary policy on financial stability, as they have significant general equilibrium effects. Results suggest that the impact of shocks is persistent and that financial shocks have higher impacts on the stability of the banking sector that real shocks.

Finally, the model is used to compare the different effects on financial stability of monetary policy and fiscal shocks and to draw policy implications.

1 Introduction

Modern financial markets are populated by agents connected by a deep web of mutual claims and obligations, creating an environment that amplifies the response of the financial system to shocks, Plosser (2009) and Yellen (2013). Early, pre-crisis, contributions such as Allen & Gale (2000) and Freixas et al. (2000) suggest that an highly interconnected financial system is more resilient to the insolvency of an individual bank as losses are spread among more creditors. More recent papers, such as Blume et al. (2013), have challenged this view arguing that interconnections may be destabilizing, allowing shocks to spread faster between agents; a similar conclusion is reached by global games models of credit connections in the spirit of Dasgupta (2004). Moreover, Acemoglu et al. (2015) has shown that the structure of the financial system is crucial to understand the spread and consequences of exogenous shocks. These conclusions are similar to the computational results by Gai & Kapadia (2010), which base their characterization of the interbank market on the empirical evidences of Caldarelli et al. (2006) and Clauset et al. (2009) between others. Departing from that literature this paper develops a full general equilibrium model with an interbank network. The advantage of this approach is that it can be used to analyze the interaction between aggregate shocks, the state of the economy and financial stability. Banks, in this model, hold at the same time claims and liabilities against each other and a network of interbanking relations arises on which contagion can be modelled in the spirit of Acemoglu et al. (2015).[1] Nesting a network structure in a DSGE setting is not straightforward as network models often exhibit chaotic properties making difficult to define a proper *steady state*. This paper solves that complexity using the moment generating function approach developed in the physics literature on diffusion, see Newman et al. (2001) and Strogatz (2001), under the assumptions that i) there are liquidity mismatches in the final loans market, such that banks need to rise liquidity from the interbank market and ii) banks interact only with a limited numbers of other banks when doing so. Specifically, banks operate, in this model, both on the interbank market (exchanging funds one another) and on a retail market (extending loans to firms). Banks need to turn to the interbank market because the demand for loans and the liquidity necessary to finance them do not arise at the same time. On that market banks trade only with a limited numbers of counterparts; this generates an endogenous interbank network with size and characteristics that evolve with the business cycle and the balance sheet of banks. The network

[1] In DSGE models with the interbank market banks are generally either landers and borrowers, see Angeloni & Faia (2009), Dib (2010) and Gertler & Kiyotaki (2010). This simplifying assumption preserves tractability but also eliminates, by construction, the possibility of contagion beyond the lander-borrower pair.

can be described using extensions of graph theory which provide estimates for the probability and size of contagion (i.e. defaults sequences). Both bank's specific characteristics and the relative position of each bank on the network are relevant for contagion. In fact, the probability that any k^{th} neighbor of a bank defaults affects the losses credit institutions are exposed to; at the same time, the balance sheet of each bank determines, for a given shock, whether it defaults or not. To model such complexity, this paper uses an extension of the moment generating function method proposed by Newman et al. (2001) to study the aggregate characteristics of the financial network and the diffusion of shocks. The other sectors of the economy, instead, follow Gertler & Karadi (2011).

The contribution of this paper is twofold. First, it provides a rigorous way to incorporate financial contagion and cascade mechanisms into a standard macro framework filling a gap in the existing literature between DSGE and contagion models. Second, the general equilibrium dimension allows to study a) how shocks generated on other sectors of the economy affect contagion risk and b) if and how the monetary and fiscal shocks can reduce that risk. Results show that financial shocks affect the contagion probability more than real shocks that, however, are long lasting. The model also highlights a trade off between macroeconomic stabilization and financial stability, with central banks that are not able to achieve financial stability with monetary policy only. Monetary policy tightenings can indeed reduce the risk of contagion, but have also severe effects on real activity. Moreover, fiscal shocks have limited effects on the stability of the financial sector in the model. The paper proceeds as follows: Section 2 Section 2 presents the model, Section 3 describe the structure of the (complex) interbank market, Section 4 shows simulations of the model under different shocks and policy specifications, in Section 5 I present conclusions. In the Appendix A and the Appendix C proofs, additional graphs and full tables are reported. Appendix B Presents an alternative configuration of the baking sector as robustness.

1.1 Related literature

This work is related to four streams of literature: financial contagion, network economics, network theory and DSGE models with financial frictions.

Financial contagion.- Since Allen & Gale (2000) a growing literature has studied how the architecture of the financial system may or not amplify contagion. Allen & Gale (2000) show how interbank relations may create fragility in response to unanticipated shocks. Additionally Shin (2008) and Shin (2009) show that securitisation enables credit expansion generating higher leverage and decreasing financial stability. More recently, Allen (2012) has shown that asset commonality between banks determines the likelihood of systemic crises while Castiglionesi et al. (2017) show that higher financial integration leads to more stability in normal times and larger interest rate spikes during crises. Dasgupta (2004) argues that financial contagion, modeled through a global game, prevents banks from perfectly ensuring on the interbank market. This paper is also closely related to Acemoglu et al. (2015), who have highlighted how the structure of the underlying financial network is not neutral for the size (and consequences) of contagion. Gai & Kapadia (2010) have reached similar conclusions using numerical simulations. Their key finding is that when networks are incomplete (i.e. banks are exposed to a limited number of counterparts) the system is fragile and a single episode can generate a cascade (a series of defaults) that may affect the entire population. If at least one first neighbor of the defaulting bank succumbs, it transmits losses to its own neighbors bringing them into the "*front line of contagion*" (Gai & Kapadia (2010)).[2] This paper extends the literature on financial contagion in two directions: first, contagion here is studied in a general equilibrium setting, with its probability and size being affected by the state of the economy and aggregate shocks; second, this paper develops a more general methodology to nest network models in general equilibrium frameworks.

Network economics.- This paper is also related to several works that try to apply complex network tools to economic problems. Most of them, such as Cabrales et al. (2017), Elliot et al. (2014), Battiston et al. (2012), Delli Gatti et al. (2010), study the propagation of shocks on a network of firms connected by credit linkages. They find that an high degree of interconnectedness allows shocks to propagate faster, making the system more fragile. This paper, in contrast, focuses on banks and on how contagion spreads on the interbank market. Another stream of the network literature focuses on the identification of the structure of interbanking connections. These works, such as Boss et al. (2004*a*), Boss et al. (2004*b*), Caldarelli et al. (2006), Soramaki (2007), Alentorn et al. (2007), Clauset et al. (2009), Cohen-Cole et al. (2011), Co-Pierre (2013) and Gabrieli & Co-Pierre (2014), use -confidential- data on interbank transactions to back out a model for the distribution of connections on the interbank market, which is critical to understand how contagion works. The general finding is that connections between banks are best modeled using a power law distribution. Power law functions have properties that capture some of the key characteristics of the interbank market structure: i) preferential attachment (i.e. banks with a link in t are more likely to have a link also in $t+1$), ii) "fat" tails (i.e. there is a relevant number of banks that

[2] On a similar topic see also Bollobas (2001), Wormald (1999), Boss et al. (2004*a*), Boss et al. (2004*b*), Caldarelli et al. (2006), Soramaki (2007), Clauset et al. (2009), Bluhm et al. (2013), Gabrieli & Co-Pierre (2014).

have a number of connections above or below the average). In the model developed hereafter those evidences seriously modeling the interbank market as a random network governed by a power law distribution.

Network theory.- This paper also adapts the methods developed by the literature on complex networks in physics and engineering to integrate a network structure into a general equilibrium model. Strogatz & Watts (1998), Callaway et al. (2000), Strogatz (2001), Newman et al. (2001) and Watts (2002) shown how real world networks can be modeled with random graphs and moment generating functions. Those papers graph theory (i.e. the set of mathematical methods to analyse a set of connections between objects) to describe the average characteristics of a graph V which distribution of connections is known.[3] The same approach can be used to study contagion as "infected" nodes are just a subsample of the entire graph V. This modelling approach takes into account both the individual characteristics (resilience, number of connections and relative position) of each object in the graph and its relative distance from all other objects. There are clear advantages of applying graph theory to banking models; however those models often exhibit chaotic properties and fractal structures which make more difficult to integrate them in a standard DSGE framework;[4] a contribution of this paper is to develop a tractable method to incorporate them into a standard monetary model in the spirit of Gertler & Karadi (2011).

DSGE models with financial friction.- This paper is obviously related to the literature on financial frictions in DSGE models. Since the global financial crisis several important contributions have focused on the interaction between financial constraints and the real economy, see for example Faia & Iliopulos (2011), Angeloni & Faia (2013), Christiano et al. (2014), Brunnermeier & Sunnikov (2014), Gertler & Kiyotaki (2015).[5] This stream of research studies the multiplicative effects of financial frictions, i.e. friction in the flow of funds between borrowers and lenders, that may arise from information asymmetries between borrowers and lenders or market power of financial intermediaries. Significant advances have been done on this front, which have lead to a better understanding on how financial frictions affect the transmission of shocks and monetary policy. This paper builds on those results incorporating into a state of the art DSGE model with financial frictions (as in Gertler & Karadi (2011)) a network model which can be used to study contagion and diffusion. The model proposed hereafter connects the literature on financial friction with that on contagion and

[3]See Vega-Redondo (2007) for an extensive review of these methods.

[4]See Mandelbrot (2004)

[5]Other notable contributions are Gerali et al. (2010), Gambacorta & Signoretti (2013), Giri (2018), Minesso Ferrari (2019).

diffusion on financial networks.

1.2 What is a cascade?

Before turning to the model, it is worth providing an intuition of the process this paper tries to analyse and the methodology used. A network is nothing else than a graph, a mathematical structure used to model pairwise relations between objects, Gross & Yellen (2003). Objects on the graphs are called nodes or vertices while their relations are described by edges connecting them.

Network analysis was originally developed in physics and engineering to trace how an impulse (i.e. energy on a grid, heat in fluids or gases, viruses between computers) spreads on a complex system which connection structure is known or can be (reasonably well) approximated by a (estimated) function. On financial markets the same tools can used to describe the web of connections between banks and how contagion spreads.[6] In this paper the objects connected in the network (graph) area banks while debt contracts are the edges linking banks with each other.

Once the network is defined it is possible to compute the probability that the default of a single node (bank) leads to a sequence of defaults and the number of nodes (banks) involved.[7] These are important measures for financial stability as they capture the probability and size of financial contagion. An important caveat is that banks do not systematically (i.e. in the steady state) default in this setting; the initial default is treated, instead, as an exogenous shock. If one bank defaults it spreads losses to its counterparties (the banks from which it has borowed) that may or may not default as well starting a contagion process. In this literature, the contagion process is called a cascade (i.e. how the impulse given to a single node spreads trough the system). The size and probability of a default sequence is affected not only by the magnitude of the initial seed[8] (the bank that originally defaulted) but also, and probably more importantly, by the relative position of the initial seed in the network.[9] Isolated banks or banks with "resilient" neighbors do not spread contagion. On the contrary the initial seed grows if and only if it is surrounded by fragile nodes. Consider the simple network of Figure 1.1.

[6] See Strogatz & Watts (1998), Strogatz (2001), Vega-Redondo (2007).

[7] The tools applied here are very similar to those used by engineers to analyze power grids or integrated circuits. Stanley (1971), Barabasi & Albert (1999), Strogatz (2001), Callaway et al. (2000), Newman et al. (2001) provide intuitions on how those methods can be applied on social networks. Those tools have also been adopted in epidemiology.

[8] So to how "large" is the institution that initially defaults.

[9] The relative position of a node in the network is given by the position and connectivity of the node and the number, position and connectivity of all its neighbors.

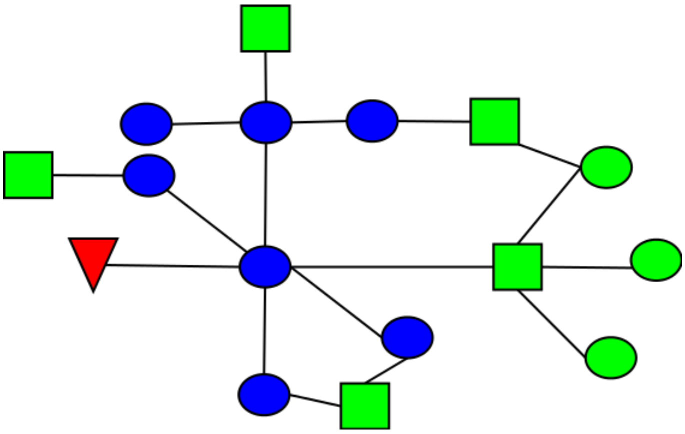

Figure 1.1: A simple network model.

Notes: The red triangle is the initial seed (i.e. the institution starting the contagion), that moves exogenously from state 0 (no default) to state 1 (default). Circles are vulnerable nodes (i.e. nodes that move to state 1 if hit by a shock) and squares are resilient nodes (i.e. nodes that never move to state 1). Green nodes are in state 0 while blue nodes are in state 1. The cascade is composed by the initial seed and all blue nodes.

The initial seed is connected to only one node. However, this node is vulnerable and is very well connected in the network, hence the initial seed can grow and infect all blue nodes in the figure. There are also vulnerable nodes that are not part of the cascade, as they are connected only to resilient nodes that protect them from contagion. This very simple example is useful to point out four characteristics of contagion that the model can capture: i) it is necessary to consider the position of each node on the network to assess its relevance; ii) diffusion processes are affected by the connectivity of any k^{th} order neighbor of the initial seed; iii) a cascade can grow through those neighbors even if it starts from a rather peripheral node; iv) resilient nodes are key to prevent contagion. The paper builds on Newman et al. (2001) and Watts (2002) to create a statistical model for the financial network.

The model takes a step forward relative to the existing literature as explicitly defines the likelihood of a bank being part of a cascade as the convolution of bank's specific characteristics (i.e. its number of links or the quality of its capital), its relative position in the network (i.e. having resilient nodes between its first and second neighbors) and aggregate shocks or macroeconomic dynamics (i.e. demand for loans, TFP, preferences). The application of network models into a DSGE framework is presented in sections 3.1 and 3.2.

2 The Model

The model follows Gertler & Karadi (2011). The economy is populated by five types of agents: households, firms, banks, government and a central bank.

Households own firms and banks, supply labor, consume, pay lump-sum taxes and save (through bank deposits). Household members are either workers -i.e. agents who supply undifferentiated labor to firms[10]- and bankers, i.e. agents who manage banks. Workers earn wages while bankers earn dividends that are transferred to the household only upon exit. Within each household there is full consumption insurance and at any moment in time the fraction $1 - f$ (f) of workers (bankers) is constant. At the end of each period, with an exogenous probability θ^B a banker stays banker for the next period and with the complementary probability "exits" and becomes a worker.[11] Finally households save through deposit accounts.[12]

The private sector of the economy is composed by wholesale producers, retailers and capital producers. Wholesale producers operate in perfect competition and produce an homogeneous undifferentiated intermediate good which is sold to retailers. They combine labor and capital in production. Capital is entirely financed by bank loans and there is no agency problem between firms and banks[13] Capital producers use a fraction of final goods and undepreciated capital as inputs to produce new capital. Retailers acquire undifferentiated goods from wholesale firms, differentiate them with negligible costs (acquiring some degree of market power following Dixit & Stiglitz (1977)) and sell them in consumption bundles on the final market. The price of final goods is sticky and can be updated in each period with probability θ_R.[14]

The financial sector of the economy is composed by a large number of banks. Each bank operates in perfect competition and is operatively divided into a wholesale branch and a retail branch. The banker's objective is to maximize end-of-period net worth.[15] As in Acemoglu et al. (2015), I assume that banks are limited in the ammount of own funds that can directly invest into projects. This is a simple way to model the fact that investment opportunities may not arise when there are internal

[10]More precisely, workers supply labor to wholesale producers.

[11]$\left(\left(1-\theta^B\right) f\right.$ is the fraction of household members who switch from banker to worker).

[12]As Gertler & Karadi (2011) households do not hold deposits at the same financial institution they control. This assumption is needed to characterize the agency problem between banks and depositors.

[13]When banks extend credit to firms they are acquiring a state contingent claim on the firm's profits.

[14]See Calvo (1983).

[15]This is equivalent to maximizing the transfer to the household in case of exit.

funds readily available to finance them. It is equivalent to assume that each period is divided into subperiods in which some banks have access to investment projects while others do not. As in Gertler & Kiyotaki (2015) the constraint generates an interbank market where banks borrow from and lend to other banks.

The retail branch provides final loans to firms using interbank credit and accumulated capital. The wholesale branch operates on the interbank market lending to and borrowing from other banks. This branch acquires loans on the interbank market when valuable investment projects arise, but the bank does not have internal funds to finance them. On the contrary, it issues interbank credit when other banks have investment projects but no liquidity available, see Acemoglu et al. (2015).[16] As a consequence, banks can borrow from and lend to other banks in the same period. This is a key characteristic of real-world bank networks and is crucial for the transmission of shocks through lending relations. In other contributions, such as Allen & Gale (2000) or Gertler & Kiyotaki (2010), the banking system is composed by a set of lending banks and a set of borrowing banks (with no intersection); banks, in other terms, are either borrowers or lenders. Under that assumption, if a borrower defaults losses do not spread beyond the direct creditors of the defaulting institution and contagion cannot start by construction (banks receiving the shock do not transmit it as they have not borrowed funds). In this model, on the contrary, contagion can spread as banks are at the same time borrowers and lenders.

I further assume that banks interact with a limited number of partners on the interbank market in each period. This constrain arises because not all banks have liquidity available at the same time, so (i.e. bank a may have funds to borrow but not when bank b needs them); when a bank demand funds, only some banks have liquidity to spear.[17] The number of borrowers (k) available to each bank is randomly drawn in each period and determines the number of (borrowing) connections a bank has. Given a distribution function from which k is drawn, it is possible to model the structure of the network with a moment generating function approach.[18] That model describes the average characteristics of the network, for example the average number of connections or the presence and size of clusters, and can be used to study the probability that the default of a single bank affects the entire banking system (i.e. a "cascade"). In this paper cascades are not systematic events, but

[16] Following Dasgupta (2004) between others, it is assumed that banks can trade only a fraction of deposits on the interbank market.

[17] An alternative justification for the constraint is that here are costs connected to exploring the network. Banks, therefore, find optimal to limit the number of partners they approach.

[18] See Newman et al. (2001), Watts (2002) and Newman (2002).

are treated as exogenous shocks to the system (i.e. one bank exogenously defaults). Hence, agents assign a very low (zero in the limit) probability that a cascade takes place. The network model, however, can be used to investigate what is the probability that the *exogenous default* of a randomly chosen bank triggers a cascade and what would be its *average* size. In this setting, the default of a credit institution might have global implications depending on the state of the economy but also on bank-specific variables. The cascade conditions, which link macro variables and the balance sheet of banks to the diffusion of shocks on the interbank market, are presented in Section 3.

An alternative approach would be that banks trade their net worth on the interbank market. this configuration of the model follows more closely the original framework by Acemoglu et al. (2015) and is explored in Appendix B.

The microeconomic literature on banking sometimes takes a different assumption to derive interbank credit linkages. Banks exchange deposits on the interbank market (instead of internal funds) to face ex-post liquidity constraints. This class of models, in the spirit of Dasgupta (2004), assume a two-period economy where banks do not accumulate net worth. In the second period households face a preference shock and a fraction of depositors withdraw deposits unexpectedly from some banks. Those credit institutions enter the interbank market to acquire liquidity. Credit linkages between financial institutions are thus engineered through deposits rather than net worth, as liquidity mismatches derive from ex-post shocks. In the original framework by Acemoglu et al. (2015), however, banks liquidity mismatches exist ex-ante: banks receive investment proposals but do not have readily available liquidity to finance them. This choice has also the advantage to link conditions on the interbank market across periods, therefore I follow as baseline assumption. The alternative approach, where connections between banks depend on deposits rather than net worth, is explored in the Appendix B. Finally, the public sector of the economy is compose by the government, which acquires final goods financing with lump-sum taxes and the central bank, which follows a Taylor-type rule.

2.1 Households

There is a continuum of identical households, indexed by $j \in [0, 1]$. Households consume, save and supply labor to firms. The preferences of a representative household are:

$$U\left(C_t, l_t\right) = \sum_{t=0}^{\infty} E_t \beta^t \left[exp(e_t^c) \ln\left(C_t - \gamma C_{t-1}\right) - \frac{\chi}{1+\varphi} l_t^{1+\varphi} \right] \tag{1}$$

where C_t denotes consumption and l_t labor at time t. There is habit formation in consumption with $\gamma \in [0,1]$ the habit parameter; χ is the weight of disutility of labor in the period utility function and φ the inverse of the Frish labor supply elasticity;[19] e_t^c is an I.I.D. preference shocks that follows an AR(1) process whose steady state value is 0. The intra-period budget constraint is:

$$C_t + D_t \leq W_t l_t + R_t^D D_{t-1} + \Pi_t \tag{2}$$

Sources of funds are: the wage bill ($W_t l_t$), with W_t real wages; returns on risk free deposits ($R_t^D D_t$) with D_t deposits and R_t^D the (real) risk free rate; Π_t collects profits (from firms and banks) net of tax paid.[20] Uses of funds are consumption (C_t) and savings (as deposits). Each household maximizes equation (1) with respect to consumption, labor supply and deposits subject to the sequence of constraints of equation (2) and non negativity constraints. First order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{e_t^c}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left[\frac{e_{t+1}^c}{C_{t+1} - \gamma C_t}\right] - \lambda_t^C = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = -\chi l_t^{\varphi} + \lambda_t^C W_t = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial D_t} = E_t\left(\lambda_{t+1}^C R_{t+1}^D\right) - \lambda_t^C \beta^{-1} = 0 \tag{5}$$

where $\left\{\lambda_t^C\right\}_{t=0}^{\infty}$ is the sequenceof Lagrangian multipliers associated to the optimization problem. λ_t^C defines a stochastic discount factor as: $E_t\left(\Lambda_{t,t+1}^C\right) = \beta^t \frac{E_t\left(\lambda_{t+1}^C\right)}{\lambda_t^C}$.

2.2 Wholesale Producers

Wholesale producers acquire capital from capital producers and labor from households to produce undifferentiated intermediate goods. They finance capital issuing securities to banks in exchange for loans at the price Q_t. Z_t are real returns on capital and each loan constitute a state contingent claim on future returns $(Z_{t+1}, (1-\delta) Z_{t+2}, (1-\delta)^2 Z_{t+3} \ldots)$, with δ the depreciation rate of capital. As the intermediate good market is perfectly competitive, the price of intermediate goods (P_t^W) equals the marginal cost. Firms produce with the technology:

$$Y_t = A_t \left(U_t \xi_t K_t\right)^{\alpha} l_t^{1-\alpha} \tag{6}$$

[19] See Frisch (1959).

[20] Notice that profits are collected from *exiting banks* and retailers.

with $\alpha \in (0,1)$, Y_t output, K_t capital, U_t the utilization rate of capital, ξ_t is a capital quality shock.[21] A_t is total factor productivity that follows an AR(1) process with steady state 1; the same AR(1) structure is assumed for ξ_t. Wholesale firms choose the level of inputs in order to maximize profits. First order conditions are:

$$\frac{\partial \mathcal{L}}{\partial U_t} = \lambda_t^f \alpha \frac{Y_t}{U_t} = \Delta' (U_t) \xi_t K_t = Z_t \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = \lambda_t^f (1-\alpha) \frac{Y_t}{l_t} = W_t \tag{8}$$

with Z_t the marginal product of capital and $\left\{\lambda_t^f\right\}_{t=0}^{\infty}$ the sequence of Lagrangian multipliers associated to the problem.

The real rate of return $\left(R_t^K\right)$ per unit of capital is the sum of the marginal product of capital (Z_t) and the value of undepreciated capital $((1-\delta)Q_t - \Delta(U_t))$ divided by the cost of capital (Q_{t-1}):[22]

$$R_t^K = \frac{\left[Z_t + (1-\delta) Q_t - \Delta (U_t)\right] \xi_t}{Q_{t-1}} \tag{9}$$

The total demand for loans (B_t) is defined as $B_t = Q_t K_{t+1}$.[23]

2.3 Retailers

Retailers acquire undifferentiated wholesale goods (at the price P_t^W) and bundle them together to produce differentiated final goods that are sold on the final market at the price P_t. Each retailer (indexed by j), faces a downward sloping demand function in a Dixit-Stiglitz setting[24] for the variety of good sold, with the CES aggregator for output $Y_t = \left[\int_0^1 Y_{j,t}^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}$. $\epsilon > 1$ is the elasticity of substitution between different varieties of final goods with the aggregate price level: $P_t = \left[\int_0^1 P_{j,t}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$. Under these assumptions, the demand function faced by a $j-th$ retailer is:

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\epsilon} Y_t \tag{10}$$

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[21] Capital utilized in production is $\xi_t K_t$. As in Merton (1973) this shock allows the introduction of an exogenous source of variation in the value of capital.

[22] Notice that returns are net of non-utilized capital $\Delta(U_t)$.

[23] From perfect competition, the interest rate on loans equals returns on capital R_t^K. Notice that in this model bank loans are closer to shares than to standard debt contracts. This approach has the merit of solving the agency problem between banks and wholesale producers.

[24] Dixit & Stiglitz (1977).

In each period retailers are able to reset prices with probability $(1-\theta_R)$. If a retailer receives the signal, she will choose the new price $P_{j,t} = P^*_{j,t}$ in order to maximize future expected profits. Alternatively prices are indexed to past inflation.[25] The first order condition is:[26]

$$\sum_{i=0}^{\infty} E_{t-1}\left\{\theta_R^i \Lambda^C_{t,t+k} Y_{j,t+i}\left[\frac{P^*_{j,t}}{P_{t+i}}\Pi^i_{k=1}\left(\pi_{t+k-1}\right)^{\theta} - \frac{\epsilon}{\epsilon-1}P^W_{t+i}\right]\right\} = 0 \tag{11}$$

2.4 Capital Good Producers

Capital good producers operate in perfect competition and combine (subject to a quadratic cost) undepreciated capital with "investment goods" to produce new capital that is sold to wholesale firms at the price Q_t.[27] The optimality condition is (with I_t new investment goods):

$$\frac{\partial \mathcal{L}}{\partial I_t} = Q_t\left[1 - \frac{\chi_I}{2}\left(x_t - 1\right)^2 + \chi_I\left(x_t - 1\right)x_t\right] + E_t\left[\Lambda^C_t \chi_I\left(x_{t+1} - 1\right)x^2_{t+1}\frac{Q_{t+1}}{Q_t}\right] = 1 \tag{12}$$

where $x_t = \frac{I_t}{I_{t-1}}$. Equation (12) is the well known Tobin's q equation,[28] that links the price of capital to its marginal cost. Finally, the law of motion of capital is: $E_t\left(K_{t+1}\right) = \left(1-\delta\right)K_t + \left[1 - \frac{\chi_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] I_t$.

2.5 Banks

Banks combine net worth, interbank loans and deposits to supply credit to firms. Each bank is managed by a banker, with the fraction of bankers in each household constant and equal to f. At the end of the period a banker may exit with probability $1-\theta_B$. If that happens, the banker transfers all remaining equity to her household as dividends.[29]

Bankers maximize the transfer to the household upon exit. Following Gertler & Karadi (2011)

[25] More formally, they solve $\max \sum_{i=0}^{\infty} E_t\left\{\theta_R^i \Lambda^C_{t,t+i}\left[\frac{P^*_{j,t}}{P_{t+i}}\Pi^i_{k=1}\left(1+\pi_{t+k-1}\right)^{\theta} - P^W_{t+i}\right] Y_{j,t+i}\right\}$, subject to the demand function, equation (10)

[26] With the optimal price level $P_t = \left[\theta_R\left(\pi^{\theta}_{t-1}P_{t-1}\right)^{1-\epsilon} + \left(1-\theta_R\right)P^{*1-\epsilon}_{j,t}\right]^{\frac{1}{1-\epsilon}}$. Since all retailers are equal, they choose the same -optimal- price, so the index can be droppend (i.e. $P^*_t = P^*_{j,t}$).

[27] Their objective function is $E_t \sum_{t=0}^{\infty} \Lambda^C_{t,t+1}\left\{Q_t I_t\left[1 - \frac{\chi_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] - \left[\left(\overline{Q_t} - Q_t\right)\left(1-\delta\right)K_{t-1}\right]\right\}$ with $\overline{Q_t}$ the price of undepreciated capital and $\frac{\chi_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2$ physical adjustments costs.

[28] See Tobin (1969). Notice that $Q_t = 1$ in the steady state.

[29] At the same time, a fraction $\theta_B f$ of workers become bankers, receiving an initial lump sum transfer from households.

and Gertler & Kiyotaki (2015), I assume that this generates an agency problem between lenders and the banker. Namely, bankers can divert a fraction Θ of funds intermediated on the retail market directly to the household, leading to a default of the bank.[30] Except for the frictions just discussed financial markets operate in perfect competition.

I depart from Gertler & Karadi (2011) assuming that banks interact on a complex network of credit relations. To model this network I incorporate the framework[31] proposed in Acemoglu et al. (2015) in a general equilibrium setting. In Acemoglu et al. (2015) two main assumptions characterize the interbank market. First, there are liquidity mismatches between financial agents. Banks cannot use their net worth to finance loans; instead they need to borrow from other credit institutions on the interbank market. This assumption captures, in a simple way, the fact that investment opportunities and the liquidity needed to finance them may not arise simultaneously. It is equivalent to assume that each period is composed by multiple subperiods. In each of them some banks have access to investment opportunities while other have liquidity. As a consequence banks exchange credit on the interbank market. Second, banks can trade only with a limited (random) number of counterparts in each period on the interbank market. This assumption captures the regularity of existing banking networks where banks have access to the entire interbank market but approach only a limited number of other credit institutions (for example because of searching costs or liquidity mismatches). See Acemoglu et al. (2015) for a detailed discussion of both assumptions.

Under these assumptions banks create a credit network of mutual claims and obligations. That network is modelled following Watts (2002) and Newman et al. (2001) and allows to quantify the likelihood that the default of a credit institution spreads to other banks. Defaults are not systematic events (i.e. they do not occur in the steady state) but are exogenous shocks. Agents, therefore, assign a very low (zero in the limit) probability that a default occurs. Nevertheless, the network models trace how contagion spreads between banks following such exogenous events. I assume that banks also differ in their ability to absorb losses when there is a default; this assumption can be relaxed without changing the main implications of the model. The following section of the paper will (extensively) elaborate on the characterization of banks. Specifically, the derivation of the interbank network is presented in section 3 under the previously described assumptions.

[30] Banks can monitor more efficiently other banks. Therefore, bankers can divert only funds intermediated on the retail market. This assumption is similar to Gertler & Kiyotaki (2010) and Gertler & Kiyotaki (2015).

[31] That is based on the results of the “coconut” model by Diamond (1982).

2.5.1 The wholesale branch

Similar to Gerali et al. (2010), the wholesale branch lends to and borrows from the interbank market. Interbank credit is then transferred to the retail branch that combines it with deposits to supply final loans to firms. At the same time, the wholesale branch issues credit on the interbank market using a fraction of deposits (Ξ) from households;[32] banks access the interbank market because there are liquidity mismatches between banks and firms (i.e. bank a does not have liquidity when firms demand credit).[33] The wholesale branch chooses the level of interbank loans offered $\left(Int_t^+\right)$ and demanded $\left(Int_t^-\right)$ to maximize the value of the branch (i.e. the discounted sum of profits) given market rates:

$$E_t(V_t^{WB}) = \sum_{i=0}^{\infty} \left(1-\theta^B\right) (\theta^B)^i \Lambda^C_{t,t+1+i} \Big[Int_t^+ R^I_{t+1} - \Xi D_t R^D_{t+1} - Int_t^- R^I_{t+1} \Big] \tag{13}$$

subject to the resource constraint $Int_t^+ \leq \Xi D_t$ and the demand of interbank loans by the retail branch $Int_t^+ = B_t - N_t$. R^I_{t+1} is the interbank rate and R_{t+1} the opportunity cost of unused capital. As there are no credit frictions on the wholesale market, the interbank market operates in perfect competitions; rates are set at the marginal cost and banks supply all funds available. First order conditions are:[34]

$$R^I_t = R_t \tag{14}$$

$$Int_t^+ = Int_t^- = \Xi D_t \tag{15}$$

These conditions translate the interbank market structure of Acemoglu et al. (2015) in a dynamic general equilibrium setting and under the assumption that banks trade a share of deposits on the interbank market. As long as equations (14) and (15) hold (i.e. the interbank rate is above or equal to the shadow cost of capital), banks supply all available liquidity on the interbank market. Finally banks trade only with a limited number of partners, k, in each period with the probability of having k partners being p_k. This is the same assumption taken by Acemoglu et al. (2015) and

[32] In this, the model follows Dasgupta (2004) between others. An alternative approach would be that banks use a fraction of banks' capital to finance interbank loans, as in Acemoglu et al. (2015); this would lead to a weaker relation between interbank risk and the banks' leverage and is explored in Appendix B.

[33] This is equivalent to assuming that each period is divided into subperiods in which some banks face higher than expected demand for loans while other lower than expected. Banks cannot divert funds intermediated on the interbank market. This is one of the two cases presented in Gertler & Kiyotaki (2015); as credit frictions are not the focus of this paper, I choose here the simplest configuration for the interbank market.

[34] Notice that these conditions are equivalent to Gerali et al. (2010) in the *absence of frictions*.

Gai & Kapadia (2010) and captures the fact that banks have access to the entire interbank network but do business only with a limited number of credit institutions. This assumption is also key to generate cascades that have general equilibrium effects. In fact, in models where banks trade with all other banks on the interbank market, losses are spread evenly between all creditors and become negligible as the interbank market grows.[35] Banks also differ in the number of defaults they can absorb, $\varphi \in [0, k]$. In section 3 it will be derived how φ depends on the bank's balance sheet, interest rates and a bank-specific component. φ can be also constant across banks without loss of generality.

2.5.2 The retail branch

The retail branch combines interbank credit and net worth to supply final loans to firms (B). Bankers maximise the value of the branch at the end of the period:

$$E_t\left(V_t^{RB}\right) = \sum_{i=0}^{\infty}\left(1-\theta^B\right)\left(\theta^B\right)^i \Lambda^C_{t,t+1+i}[R^K_{t+1+i}B_{t+i} + R_{t+1+i}N_{t+i} - R^I_{t+1+i}Int^-_{t+i}] \tag{16}$$

As in Gertler & Karadi (2011) and Gertler & Kiyotaki (2015), at the beginning of the period bankers can also divert a fraction Θ of intermediated assets to households. The banker decides *not to* divert assets if the expected end-of-period value of the branch is larger than the value of divertable assets:

$$E\left(V_t\right) \geq \Theta B_t \tag{17}$$

The (recursive) solution for the optimal value of the branch is:

$$E\left(V_t\right) = k_t^1 Int_t^- + k_t^2 B_t \tag{18}$$

with k_t^1 and k_t^2 defined as:

$$k_t^1 = E_t\left[\left(1-\theta^B\right) + \Lambda^C_{t,t+1}\theta^B \frac{N_{t+1}}{N}_t k^1_{t+1}\right] \tag{19}$$

[35]Each bank divides evenly its desired amount of interbank loans between all possible borrowers. Banks differ only in their position on the credit network and in their haircut rate in case of default. As the expected default probability is zero, all banks are ex-ante equal (so each of them has the same "risk").

$$k_t^2 = E_t \left[\left(1-\theta^B\right) \Lambda_{t,t+1}^C \theta^B \left(R_{t+1}^K - R_{t+1}^I\right) + \Lambda_{t,t+1}^C \theta^B \frac{B_{t+1}}{B_t} k_{t+1}^2 \right] \tag{20}$$

The incentive compatibility constraint can be written as $k_t^1 Int_t^- + k_t^2 B_t \geq \Theta B_t$ so that, using the budget constraint of the retail branch[36], equation (17) becomes:

$$B_t = \phi_t N_t \tag{21}$$

$\phi_t^L = \frac{k_t^2}{\Theta - k_t^1}$ is the ratio between net worth and interbank funds (similar to the leverage ratio in Gertler & Karadi (2011)). The constraint given by equation (17) is locally biding around the steady state and determines the spread between the rental rate of capital and the risk free rate $(R^K - R^D)$ as a function of the degree of financial frictions.[37]

End of period banks' capital (N_t^{EP}) is equal to the beginning of period capital (N_t) plus profits from both branches:[38]

$$E_t\left(N_{t+1}^{EP}\right) = \left(R_{t+1}^K - R_{t+1}^D\right) B_t + R_{t+1}^D N_t \tag{22}$$

The total net worth at the beginning of the next period is the sum of net worth of surviving banks $(\theta^B N_{t+1}^{EP})$ and the initial transfer from households to new banks:[39]

$$E_t(N_{t+1}) = E_t(\theta^B N_{t+1}^{EP} + \frac{\omega}{1-\theta^B} N_{t+1}^{BO} + e_{t+1}^B) \tag{23}$$

e_{t+1}^B is a shock on the bank's capital that follows an AR(1) process and has steady state value of 0. Notice that in this model banks do not systematically default as loans to firms bear no risk. Defaults starting contagion are, instead, exogenous shocks to the banking sector.

[36] That is $Int_t^- + D_t = B_t$.

[37] Gertler & Karadi (2011) provides a proof for this statement.

[38] Given the assumptions on bankers' behaviour, bankers optimally accumulate profits (if they do not exit) in each period. See Bernanke et al. (1999) and Gertler & Kiyotaki (2010) for a detailed discussion.

[39] Houshold transfer a fraction $\frac{\omega}{1-\theta^B}$ of the equity of exiting banks to newly formed banks.

2.6 Equilibrium

Equilibrium relations complete the model. The supply-demand identity must hold on the good market:

$$Y_t = C_t + I_t + G_t + \Phi_t \tag{24}$$

where Φ_t gathers additional costs for the economy (adjustment costs in capital goods production and frictions) and G_t is public expenditure, which is assumed to be exogenous and to follow an AR(1) process. The central bank sets nominal rates following a Taylor-type rule:[40]

$$ln(R_t^n) = (1-\rho)\left[ln(\overline{R^n}) + \psi_\pi ln(\pi_t) + \psi_y(ln(Y_t) - ln(Y^{ss}))\right] + \rho ln(R_{t-1}^n) + e_t^R \tag{25}$$

where R_t^n is the net nominal interest rate, $\overline{R^n}$ its steady state value and e_t^R is a monetary policy shock that follows an AR(1) process with steady state 0. The real rate is defined by the Fisher relation $(1 + R_t^n = E_t\left(R_{t+1}\pi_{t+1}\right))$.[41]

3 The Interbank Market

Banks are connected by mutual claims and obligations. As banks trade with only a fraction of all available counterparts, the default of one bank might have effects on the entire population depending on the bank's position and the characteristics of its neighbours. To model how contagion spreads, a network model is derived under the assumption outlined in the previous section. The characteristics of the resulting network are time-varying and depend on aggregate macro and financial variables as well as on the balance sheet of banks.

3.1 A General Model of Transmission

A common method in physics and engineering to model diffusion on random graphs (which are sets of interconnected entities) is the so-called moment generating functions (MGF) approach.[42] In a nutshell, MGF allow to summarize the average characteristics of a grapth exploiting the proba-

[40] Taylor & Woodford (1999).

[41] See Fisher (1896).

[42] I make use of the methods proposed by Strogatz & Watts (1998), Callaway et al. (2000), Strogatz (2001), Newman et al. (2001) and Watts (2002). This method can be applied to any random graph independently on the underlying distributions.

bility distribution (PDF) of its nodes' connections. If a PDF for the probability that a node has k connections is available, through moment generating functions the characteristics of the entire network can be derived analytcally. This approach is particularly appealing as it allows to take into consideration all k levels of connections between nodes (banks) in the graph. The probability that bank a is part of a default sequence depends on the institutions with which it is directly connected (say b and c), but also on how b and c are connected themselves and how all their neighbors are. This process goes on to the infinite; Strogatz (2001) and Newman et al. (2001) have developed a computational method to solve such complexity and analytically derive the probability and size of a cascade for a random network with a given PDF for the nodes' links.

Specifically, consider a random graph composed by V nodes (in this contex banks) and d edges (credit linkages). The probability of a node having k edges is defined as p_k.[43] Each node has two possible states: 0 (inactive) to state 1 (active); it moves from 0 to 1 if at least a fraction ϕ of its k neighbors are in state 1.[44] ϕ is a parameter specific to each node and is drawn from a distribution $f(\phi)$ that is normalized such that $\int_0^1 f(\phi)\, d\phi = 1$. Notice that this assumption can be relaxed by assigning to each node the same value of ϕ; in this case node differs only in the number of connections.

A cascade is defined as the number of nodes that move from state 0 to state 1 if a random node (called the *initial seed*) exogenously shifts to state 1 (if nodes are banks that is a default). The model identifies: i) the probability that a global cascade is triggered by a single node, where a global cascade is defined as a cascade that occupies a finite fraction of the total number of nodes; ii) the size of the cascade if it takes place. In a banking network that would be: i) the probability of contagion in case bank chosen at random defaults, ii) the number of banks affected. Most importantly, default sequences are started by the exogenous default of one bank; banks do not default systematically in this model. From the properties of random graphs[45] follows that in large random graphs with small initial seeds the local neighboring of an initial seed does not contain cycles,[46] therefore no vertex neighbor to an initial seed is adjacent to more than one seed. This property is important to pin down the diffusion process. Another important property of large random graphs is that they can be regarded as pure branching structures, therefore each subcluster *below the transition size* may

[43] In this model k is the number of conterparties with which each bank trades. On large networks p_k is also the share of banks that have exactly k partners.

[44] This is equivalent to say that a node moves to state 1 if receives enough shocks (ϕ) from its neighbors.

[45] The interested reader may refer to Newman et al. (2001), Watts (2002) or Vega-Redondo (2007).

[46] See Watts (2002) and Gross & Yellen (2003), this is asymptotically true for random graphs as the number of vertices grows. See Appendix A.2 for an intuition.

be regarded as independent.[47]

Under these conditions, an initial seed can diffuse through the network if and only if at least one of its initial neighbors has a threshold $\phi \leq \frac{1}{k}$ or $k \leq \bar{k} = \frac{1}{\phi}$; a vertex that meets that condition is called vulnerable as it might be part of the cascade. The existence of a cascade depends not only on the magnitude of the initial shock but also, and maybe more importantly, on the position of the initial seed(s). A seed that is isolated does not threaten the whole system even if large in magnitude. On the contrary, a small seed located in a very "central" position may expand through the entire graph.

Define the probability of a vertex with k neighbors to be vulnerable as $\rho_k = P\left[\phi \leq \frac{1}{k}\right]$.[48] The moment generating function of the vulnerable vertex degree is:[49]

$$G_0(x) = \sum_{k=0}^{\infty} \rho_k p_k x^k \tag{26}$$

the moments of this function define the moments of the vulnerable cluster (i.e. the group of nodes -banks- that are affected by the initial seed).[50] The vulnerable fraction of the population (i.e. the probability that a bank triggers a cascade when defaults) is simply given by $P_v = G_0(1)$.[51] Notice that a node is vulnerable depending *at the same time* on its own value of ϕ and on the number and vulnerability of its neighbors. The combination of the two defines weather a vertex (bank) in the model is part of the vulnerable cluster. The average degree of vulnerable vertices (i.e. how many connections vulnerable vertices have *on average*) is $z_v = G_0'(1)$. A third element to evaluate is the degree distribution of a vulnerable randomly chosen vertex a that is a neighbor of the initial (active) vertex b. The larger is the degree of a the more likely it can spread contagion from b. The probability of choosing a is proportional to kp_k and the corresponding, well-behaving, moment generating function is:

$$G_1(x) = \frac{\sum_{k=0}^{\infty} \rho_k k p_k x^{k-1}}{\sum_{k=0}^{\infty} k p_k} \tag{27}$$

the numerator is the first derivative of equation (26) while the denominator is simply the average

[47] A cluster is said to be *at the transition size* if a shock starting there can percolated to the entire network. See Newman et al. (2001) and Watts (2002) for proofs.

[48] Notice that for $k = 0$ $\rho_k = 1$. Additionally ρ_k constant is just a special case of this model.

[49] The interested reader may refer to Newman et al. (2001), Strogatz (2001), Watts (2002) and Vega-Redondo (2007) for an extensive theory.

[50] The subscript 0 defines the moment generating function of a node, 1 that of its first neighbor, 2 that of its second neighbor and so on.

[51] An advantage of this method is that analysis are easily scalable, $P_v = G_0(2)$, for example, is the probability that a cascade starts when two nodes move to state 1.

degree of the whole network (z). Therefore, it can be simplified into:

$$G_1(x) = \frac{G_0'(x)}{z} \tag{28}$$

Finally, define q_n as the probability that a randomly chosen vertex belongs to a vulnerable cluster of size n, and r_n as the corresponding probability for a neighbor of the initial vertex. The moment generating functions for q_n and r_n (($H_0(x)$, $H_1(x)$)) are:

$$H_0(x) = \sum_{n=0}^{\infty} q_n x^n \; and \; H_1(x) = \sum_{n=0}^{\infty} r_n x^n \tag{29}$$

Self-consistency conditions[52] call for:

$$H_1(x) = [1 - G_1(1)] + xG_1[H_1(x)] \tag{30}$$

allowing to derive $H_0(x)$ as:

$$H_0(x) = [1 - G_0(1)] + xG_0[H_1(x)] \tag{31}$$

$[1 - G_0(1)]$ is the probability that a chosen vertex is not vulnerable, $xG_0[H_1(x)]$ takes into account the size of the vulnerable cluster. From the definition of $H_0(x)$ it is possible to compute all moments of the vulnerable cluster, in particular its average size $\|n\| = H_0'(1)$. Using equations (30) and (31):

$$\|n\| = G_0(1) + \frac{[G_0'(1)]^2}{z - G_0''(1)} = P_v + \frac{z_v^2}{(z - G_0''(1))} \tag{32}$$

which diverges ($\|n\| \to \infty$) when:

$$G_0''(1) = \sum_k k(k-1)\rho_k p_k = z \tag{33}$$

See Appendix A for an extensive proof. $\|n\|$, the average vulnerable cluster size, defines the average number of nodes that participate in the cascade if it takes place. P_v is the vulnerable fraction of the population that, on large networks, approximates the probability that a single node is vulnerable

[52]See Newman et al. (2001) and Strogatz (2001). Self-consistency allows to approximate the distribution of a random vector X by a random vector Y whose structure is less complex without significant loss of information. In particular we can construct a self-consistent approximation of X dividing X into subsamples and defining Y as a random variable with values the means of each subset. Notice that a (random) search process on a graph can be regarded as a pure martingale process. See Tarpey & Flury (1996) for more details.

(i.e. it might trigger a cascade moving exogenously from state 0 to 1). In this model if a node (bank) is eliminated at random, it starts a cascade (default sequence) with probability P_v that has, on average, size $\|n\|$. Notice that these measures depend on the way in which nodes are connected (described by p_k) and on the resilience of each single node (ϕ).[53]

Equation (32) is generally known as the *cascade condition*. It states that whenever $G_0''(1) < z$ all vulnerable clusters in the network are small; in other words the initial shock is unable to percolate through the whole system. On the contrary, when $G_0''(1) = z$ a percolating vulnerable cluster arises; through it an initial seed may spread to the whole system.[54] Equation (33) is also known as *phase transition* and as $k(k-1)$ is monotonically increasing in k, while ρ_k is monotonically decreasing in it, has two solutions or none. With two solutions there should be, according to Watts (2002), an interval in z in which cascades can occur. Appendix A.5 provides an overview of the properties of the network under analysis. In Appendix A.6 the interested reader may find a longer discussion.

3.2 Application to the Model

It is straightforward to identify nodes as banks on the interbank market, the initial seed as a defaulting bank and the parameter ϕ as the maximum number of defaults that a bank can absorb without defaulting itself. A "global cascade" is nothing else than contagion on the interbank market and its probability and size a good measure for systemic risk.[55] Equation (26) is the probability that the default of a randomly chosen bank triggers a sequence of episodes. The lower is P_v the less likely a cascade occurs, the more stable is the system. At the same time $\| n \|$, the cascade size, defines the magnitude of the default sequence and the severity of the episode; p_k is the distribution from which is drawn the number of counterparties with which each bank trades at each time t.[56]

This framework has two advantages: first it integrates, in a rigorous way, a model to study diffusion on the interbank network into a standard DSGE model, being, to the best of my knowledge, the first paper on this topic. Second, the position of a bank on the network (i.e. the number and characteristics of all its k^{th} order neighbors) defines the probability of being part of a cascade and the relative position of all banks defines the cascade's size. To implement the diffusion model it is

[53] In the following p_k is chosen to match the empirical literature on interbank networks.

[54] On this topic see Stanley (1971), Bollobas (2001) and Stauffer & Ahrony (1991).

[55] Cascade probability and size define, in terms of macroprudential policy, the probability of contagion spreading from one defaulting institution to the system and the *expected* number of defaulting banks involved.

[56] That number is assumed to be I.I.D between time and banks.

necessary to choose a probability distribution for p_k and then evaluate equations (26) to (33). A simple possibility would be to follow the so called small-world model by Strogatz & Watts (1998) based on Erdos & Renyi (1959) and Erdos & Renyi (1960) assuming a Poisson distribution for p_k. Results with this choice, however, fail to describe real world networks, most of which exhibit power law properties.[57] In particular, there are sound and coherent evidences that the structure of the interbank market network is described by a power law distribution, between them: Boss et al. (2004*a*), Boss et al. (2004*b*), Caldarelli et al. (2006), Soramaki (2007), Allen & Babus (2007), Alentorn et al. (2007), Clauset et al. (2009), Cohen-Cole et al. (2011), Co-Pierre (2013) and Gabrieli & Co-Pierre (2014). Following those results p_k is assumed to have a power law distribution.[58]

A power law is bounded in the interval $[1, \infty]$[59] and captures some of the key characteristics of the interbank market (as well as of other real world networks): i) preferential attachment,[60] ii) "fat" tails.[61] Define p_k as:[62]

$$p_k = Lk^{-\widetilde{\gamma}} \tag{34}$$

where $L = \zeta(\widetilde{\gamma})$ is a standardization parameter, ζ being the Reinmann ζ-function. The parameter $\widetilde{\gamma} > 2$[63] defines the degree of "concentration" in the system; as $\widetilde{\gamma}$ grows, the probability associated to large k decreases, resulting in networks with less "well-connected" nodes (banks). The deriving well-behaved moment generating function (i.e $G_0(x) = \sum_k p_k x^k$) is:

$$G_0(x) = \frac{Li_{\widetilde{\gamma}}(x)}{\zeta(\widetilde{\gamma})} \tag{35}$$

[57] See Barabasi & Albert (1999).

[58] For a detailed description of the properties of random graphs generated by this distribution see Newman et al. (2001), Aiello et al. (2001), Strogatz (2001), ben Avraham et al. (2002) and Vega-Redondo (2007).

[59] This fits perfectly with the assumptions of the model. In fact, in the model banks need at least one counterparty to operate. If k was 0, than a bank would not participate on the market and its supply of loans would be equal to deposits.

[60] In the literature on random graphs preferential attachment is a characteristic of networks where agents with more links are more likely to form new ones. This characteristic is embodied by power law distributions while it is missing in networks generated by normal distrubutions. In the terms of the interbank market, it means that larger banks tend to have more connections that small banks. This is consistent with empirical evidences on the the banking literature and the literature on interbank networks (see Caldarelli et al. (2006) and Boss et al. (2004*a*)).

[61] Graphs characterized by power law distributions have typically a significant number of nodes located in the tails of the degree distribution contrary to graphs described by normal distributions. This means that there many banks populating the upper tail of the distribution -i.e. being more connected- that are systemically important institutions. Therefore, the simple average number of connection is not informative and it is necessary to rely on additional statistics to describe the network structure. The interested reader can, again, rely on Newman et al. (2001).

[62] Notice that in this case the power-law does not exhibit an exponential cutoff. A possible extension of the model would be to incorporate it.

[63] See Newman et al. (2001), ben Avraham et al. (2002), Watts (2002) and Vega-Redondo (2007). If $\widetilde{\gamma} \leq 2$ the first derivative of $G_1(x)$ is not defined.

Equation (35) has the main drawback of exhibiting fractal and chaotic properties.[64] Fractals have not always closed form solutions and are characterized by chaotic behavior. In this case, $Li_{\widetilde{\gamma}}(x)$, the $\widetilde{\gamma}^{th}$ order polylogarithm of x, is chaotic and has closed-form solution only for some specific choice of parameters. To incorporate the cascade mechanism into the DSGE framework it is necessary to derive a more tractable solution. Recall the definition of a definite integral as the limit of a Riemann sum[65] and that the number of agents in the model is very large.[66] As in Vega-Redondo (2007) it is possible to approximate the discrete distribution to its continuous counterpart. Given an interval $[a, b]$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \tag{36}$$

with $x_i = a + i\frac{b-a}{n}$. In the case of the MGF $\sum_k f(k) = G_0(x) = \sum_{k=0}^{\infty} \rho_k p_k x^k$, with $\frac{b-a}{n} = 1$. n, the upper bound of the interval, is the maximum number of loans (edges) that a bank may have. As bank, in the model, cannot lend less than one unit of credit per counterpart, n equals Int_t^+. Therefore it is straightforward to show that $a = 0$ and $b = n$.[67] As the maximum number of connections is a function of the total ammount of interbank loans, the network structure changes following developments on the interbank market. Rewriting equation (26) we have that:[68]

$$G_0(1) = \int_1^{Int_t} \rho_k L k^{-\widetilde{\gamma}} x^k dk \tag{37}$$

with $\rho_k = P\left[\phi \leq \frac{1}{k}\right]$ and $\phi = f(V_t^e)$. Turning to ϕ, it can be proved[69] that $\phi = \frac{V_t}{Int_t}$ with V_t the -absolute- maximum amount of funds that a bank can lose without defaulting. The definition of V_t derives from the budget of a single bank:

$$V_t = \frac{R_t^K B_t - R_t^D D_t - R_t^I Int_t}{R_t^I} \tag{38}$$

[64] See Falconer (2003) and Mandelbrot (2004).

[65] See Krantz (2005).

[66] Notice that in this model there is a large number of banks to ensure perfect competition.

[67] Consider the moment generating function $G_0(x) = \sum_{k=0}^{\infty} \rho_k p_k x^k = \sum_{i=1}^{n} f(x_i)$. Additionally $x_i = k$, as k is the argument of the MGF G_0. This approximation holds when n tends to a large number. A similar method is used by Newman et al. (2001) and Vega-Redondo (2007).

[68] Notice that the power law distribution is defined over the interval $[1, \infty]$.

[69] See Appendix A.

This relationship reinforces the connection between the network structure and financial variables and also links the network to real variables (through real returns, that depend directly on K and Y). Assume, finally, that banks face an haircut on the value of their assets when they are forced to sell them during crisis. The haircut is specific to each bank reflecting bank-specific conditions.[70] This assumption is in line with empirical results and introduce heterogeneity *during crisis* between banks.[71] It can be relaxed without changing the main results of the model.[72]

To model heterogeneity across banks during distress, assume that V_t is affected by a bank specific random shock η_t^i that follows a uniform distribution over $[0, 1]$ and is I.I.D. through banks (i) and time (t). η_t^i measures the haircut on assets that a specific banks face during a cascade.[73] Each bank is able to use only:

$$V_t^e = \eta_t^i V_t \tag{39}$$

The limit case of $\eta_t^i = 1$ for $\forall t$ and i is a special case of the model under consideration, with a much simpler network structure (i.e. ϕ is constant for all banks, only connectivity determines whether they are vulnerable). $P\left[\phi \leq \frac{1}{k}\right]$ (the probability of being vulnerable) is defined as:

$$P\left[\phi \leq \frac{1}{k}\right] = P\left[\frac{\left(V_t^e = \eta_t^i V_t\right)}{Int_t} \leq \frac{1}{k}\right] = P\left[\eta_t^i \leq \frac{Int_t}{kV_t}\right] \tag{40}$$

Therefore, equation (37) becomes:

$$G_0\left(1\right) = \int_1^{Int_t} \Omega_t L k^{-\widetilde{\gamma}} x^k dk \tag{41}$$

with $\Omega_t \equiv P\left[\eta_t^i \leq \frac{Int_t}{kV_t}\right]$. The probability of a cascade is affected not only by the total amount of credit exchanged between banks but also by a bank-specific component. Notice that the resilience of a bank to external shocks depends both on the realization of η and on the numbers of connections k; both characteristics are important to determine if the bank belongs or not to a vulnerable cluster.

[70] It takes into account the specific managerial ability of some banks or the specificity of the assets in which they have invested.

[71] See for example, Angeloni & Faia (2009) and Altunbas et al. (2017).

[72] In that case, in fact, ϕ would be constant across all banks and P_v would depend only on the connectivity of each bank.

[73] See Christopher (1991), Coval & Stafford (2005), Brunnermeier (2009) and Duarte & Eisenbach (2013).

From equation (41) all relevant network variables can be computed:

$$z_t = \int_1^{Int_t} Lk^{-\widetilde{\gamma}} x^k dk \tag{42}$$

$$G'_{0,t}(x) = \int_1^{Int_t} \Omega_t Lk^{-\widetilde{\gamma}} k x^{k-1} dk \tag{43}$$

$$G''_{0,t}(x) = \int_1^{Int_t} \Omega_t Lk^{-\widetilde{\gamma}} k (k-1) x^{k-2} dk \tag{44}$$

These equations are sufficient to describe the aggregate characteristics of the interbank network and link -through interbank loans, bank capital and returns on assets- the economy to the network structure. They are the result of individual decisions of banks and of the collective actions of the banking system.[74] Network variables, in this model, are defined by: i) the state of the economy (i.e. demand for loans, returns on capital); ii) financial and bank specific conditions (capitalization of banks, volume of trade on the interbank market) and iii) the network topology (i.e. the relative position of each bank vis-*á*-vis each oterh bank). Equation (26) describes the probability that a defaulting bank triggers a cascade. It gives the probability that a bank a is connected to at least one fragile neighbor b; if a defaults a cascade is triggered (i.e. at least another bank -b- defaults). Generally, the more a network is sparse (the larger is $\widetilde{\gamma}$), the more likely a randomly chosen bank starts a cascade if it defaults.[75] Equation (32) defines the expected number of institutions involved in a cascade and is the result of a complex convolution of the parameters of the network; depends, quite intuitively, positively on the probability of being part of a vulnerable cluster (P_v) and on the average degree of vulnerable vertices.[76] On the contrary $\parallel n \parallel$ depends negatively on the whole network average connectivity z and positively on $G''_0(x)$. Their combination leads to the so called "phase transition equation" (33).[77] This equation defines the conditions at which the entire system is affected by the initial seed. Similarly to previous contributions on financial networks, that condition

[74]Notice that each banker cannot computed her own exposure to risk. In fact, to compute her own exposure to risk a banker should know her own set of neighbors on the interbank market, their characteristics and the set of connections (and characteristics) of any k^{th} order neighbor.

[75]The larger is $\widetilde{\gamma}$ the lower is the number of highly connected agents. The resulting network, therefore, is more sparse in the sense that there are relatively more agents less connected between them. Additionally $\widetilde{\gamma}$ must fall in the interval $[2, 5]$, see Appendix A.6.

[76]The more a vulnerable vertex is connected, in fact, the easier it can be "infected" and spread contagion to other vertices.

[77]See Appendix A.6 for more details.

is very rarely met, but the effects of a phase transition are potentially devastating.[78]

4 Simulation Exercises

The model is calibrated following Gertler & Karadi (2011) and Friedman & Woodford (2011). The discount parameter β is set to 0.99, γ and φ to 0.815 and 0.276 respectively. The weight of labor in the utility function (χ) is 3.409. The share of capital in the production function, α, is calibrated to 0.33. δ and ϵ, the depreciation rate of capital and the elasticity of demand, take the values of 0.025 and 4.167. The Calvo parameter is set to 0.779 and χ_I to 12.[79] The key network parameter ($\widetilde{\gamma}$) is initialized to 2.5 following the evidences of a robust empirical literature on social networks.[80] The monetary policy function parameters (ϱ, ψ_π and ψ_y) are calibrated to 0.8, 1.5 and 0.5 respectively. Finally, the share of government consumption on total output is equal to 0.2 in the steady state and the autocorrelation coefficient of each shock is set to 0.5.

4.1 Static simulations

Before analysing the dynamic response of endogenous variables to shocks, it is helpful to investigate the properties of the interbank network and in particular how the configuration of the network in the equilibrium changes when the leverage of the banking sector rises or the spread between safe and risky asset widens. This can clarify the mechanisms of transmissions between financial variables, banks' balance sheets and the contagion probability. Appendix A.5 and Appendix A.6 present a smilar, partial equilibrium, analysis of cascades. Figure 4.1 shows the evolution of the equilibrium contagion probability at different levels of leverage keeping all other variables constant.[81] When leverage (φ_t^L) is at the steady state, P_v is close to 0.13%, which is a reasonable value considering historical averages.[82] If it rises, banks also issue more interbank credit to finance loans and, hence, the probability of contagion *on the interbank market* increases. When deposits rise relative to bank

[78] See for example Acemoglu et al. (2015). Appendix A.6 provides a more extensive analysis.

[79] Following in this case Dib (2010).

[80] Between many contributions: Bollobas (2001), Wormald (1999), Boss et al. (2004*a*), Boss et al. (2004*b*), Caldarelli et al. (2006), Soramaki (2007), Clauset et al. (2009), Bluhm et al. (2013), Gabrieli & Co-Pierre (2014).

[81] φ_t^L defines leverage in this model, as it captures the ratio between deposits and net worth of banks.

[82] Data from the US Federal Deposit Insurance Corporation show that, excluding the Global Financial Crisis, the historical annual share of defaults of US banks is about 0.16. See Federal Deposit Insurance Corporation, Failures of all Institutions for the United States and Other Areas [BKFTTLA641N], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/BKFTTLA641N, January 24, 2020.

net worth there is more trade on the interbank market and, hence, banks become interconnected ($G_0'(1)$ raises). As a result, the share of banks involved in a cascade ($||n||$) increases as well.[83]

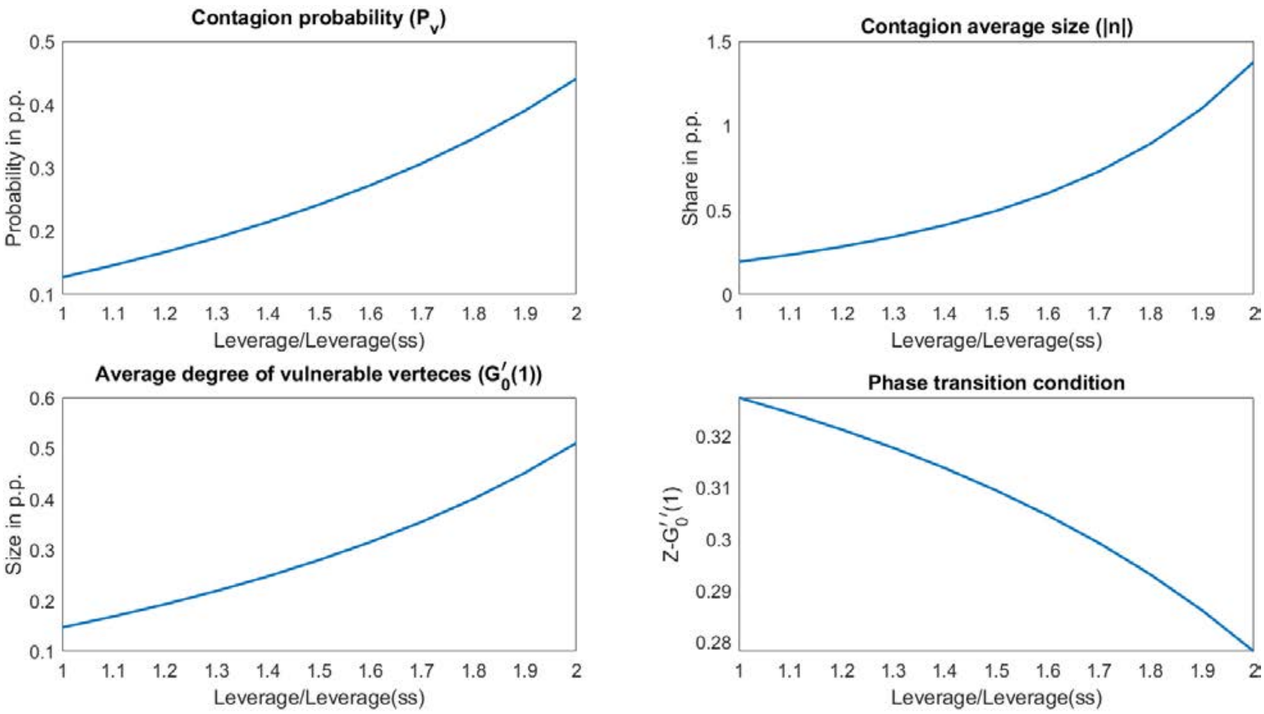

Figure 4.1: Steady state evolution of key network variables changing the equilibrium leverage of financial intermediaries.

The probability and size of contagion increase when the spreads on risky assets widens ($R^K - R$), see Figure 4.2. If spreads increase there are two opposite forces that affect the structure of the interbank network. The first channel derives from returns on final loans. If R^K rises relative to R, banks have larger profits and increase the amount of losses they can absorb without defaulting (V). The second is the demand for final loans. When returns on capital (R^K) increase entrepreneurs demand more loans as investment projects are more profitable. This leads to an expansion of credit and a rise in the demand for interbank funds, which make banks more interconnected and, hence, the risk of contagion rises. In this model the second channel prevails as the expansion in credit linkages between banks overcomes the benefits of a more resilient bank net worth.

Figures B.I and B.II in the Appendix B report the steady state value of key network variables when banks finance interbank credit using their net worth. In that case rising leverage decrease

[83]In Acemoglu et al. (2015) the contagion probability responds to the net worth of banks but it is neutral the heterogeneity in size or leverage among banks. This is a direct consequences of the assumption that banks are equal *any variation in the fragility of the system is due to the financial network's structure.* As a consequence, if banks trade net worth on the interbank market, the contagion probability and size *decrease* when leverage increase; the result also hold in this model as shown in 4.1. See Acemoglu et al. (2015), pp 565.

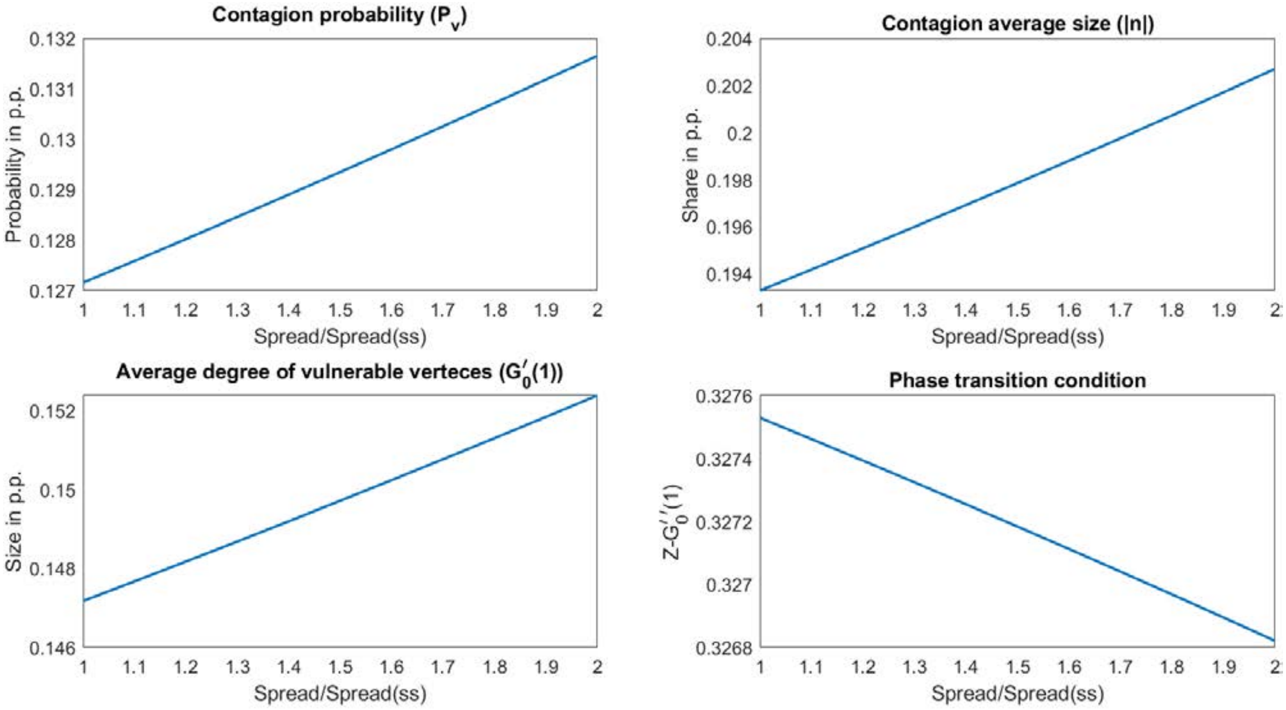

Figure 4.2: Steady state evolution of key network variables changing the equilibrium spread $R^K - R$.

both the probability and the size of contagion (as by construction more leverage reduces net worth over deposits and hence availability of interbank funds, this is similar to Acemoglu et al. (2015)). Changes in spreads, instead, have similar effects to the baseline framework.

4.2 Impulse Response to Standard Shocks

This section presented the effects of a 1% positive exogenous shock on the interest rate (i.e. a contractionary monetary policy surprise), the TFP, consumption preferences, government spending, capital quality and banks' net worth. In this regard, the model resembles standard New Keynesian dynamics for output, consumption and employment.[84] As these results are well documented in the literature, only the IRFs for the TFP and the monetary policy shocks are reported. Appendix C present the responses ot other shocks.

Output responds positively to a TFP shock and negatively to a monetary policy tightening. As expected, inflation responds negatively to both TFP and monetary policy shocks as a result of lower production costs generated by the TFP shock and a fall in aggregate demand induced by the

[84]These results are in line with previous works such as: Angeloni & Faia (2009), Christiano et al. (2010), Gerali et al. (2010), and Gertler & Karadi (2011).

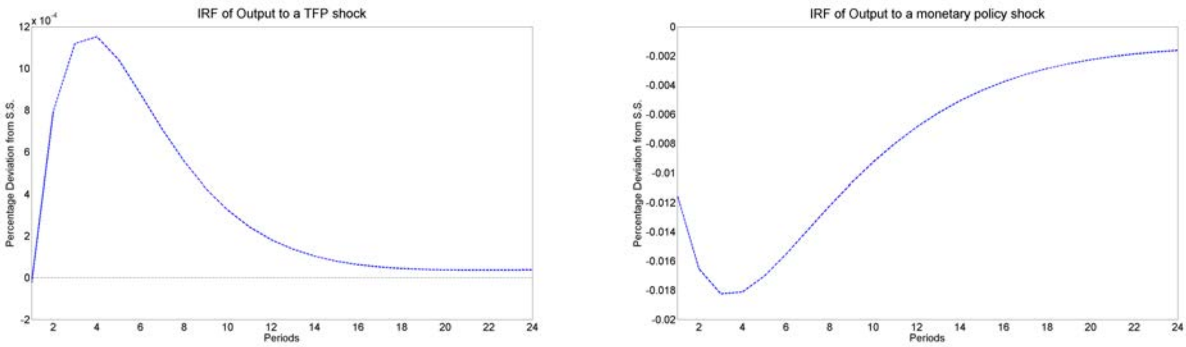

Figure 4.3: Output response to TFP and monetary policy shocks.

increased interest rate. Total loans increase after a positive TFP and decrease after a contractionary monetary policy shock. This is due to higher (lower) demand and a lower (higher) cost of funding for banks. Interbank loans, Figure 4.4, follow a similar pattern.

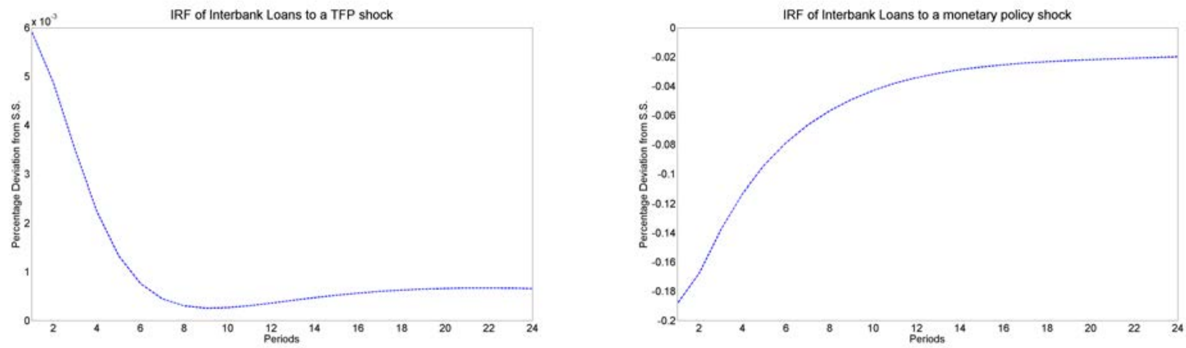

Figure 4.4: Interbank loans response to TFP and monetary policy shocks.

Finally, the central bank conducts an easing policy after a positive TFP shock as the drop in inflation dominates the increase in output.

4.3 Network Impulse Responses

We now leave the safe shores of standard IRFs to enter the uncharted territory of the interbank network structure. This exercise shows how contagion becomes more likely and its size increases after exogenous macroeconomic shocks.[85] In this context contagion is defined as the probability that the default of one bank triggers a sequence of defaults. P_v and $\parallel n \parallel$ show what would happen

[85] Although there is a wide network literature, grown faster in the last years, on financial markets (such as Gai & Kapadia (2010), Cohen-Cole et al. (2011), Battiston et al. (2013), Co-Pierre (2013), Bluhm et al. (2013),Gabrieli & Co-Pierre (2014), Capponi & Chen (2015)) few models analyze them in a general equilibrium set up assessing the evolution of banking relations over the business cycle.

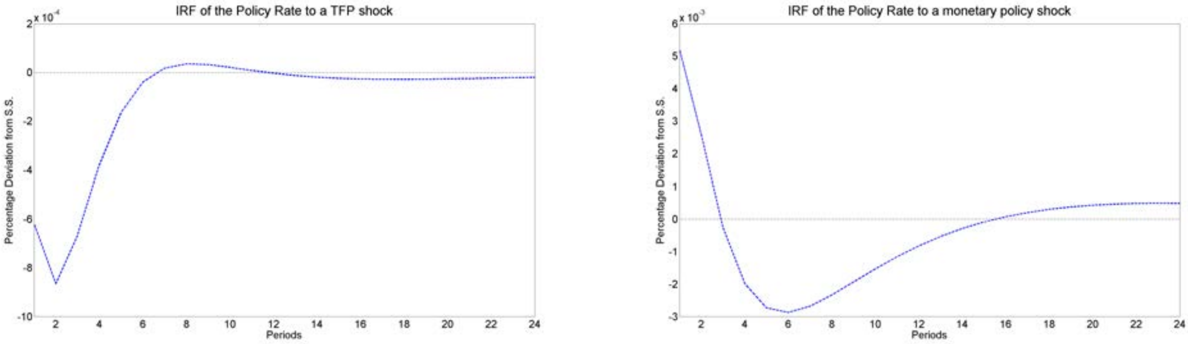

Figure 4.5: Policy rate response to TFP and monetary policy shocks.

if a bank exogenously defaults at any point in time. Recall that in this model defaults *do not occur systematically*, but are defined as exogenous shocks (i.e. one node, randomly chosen, moves from state 0 to 1).

The first variable to analyze is the share of banks belonging to a vulnerable cluster, P_v, which approximates -on large networks- the probability of a single bank being vulnerable and able to trigger cascade. According to equations (26), (38) and (39) , P_v is influenced not only by the size of the interbank market, but also by interest rates and by the propensity to save of consumers. Therefore, the interaction between financial and real markets are complex and not completely predictable a priori.

Financial shocks (monetary policy, capital quality and banks' net worth) have a stronger impact on P_v than real shocks (TFP, government spending and consumer preference). There are two reasons for that. On one hand P_v depends directly from financial variables and only indirectly from real ones. Their impact is mediated through the frictions of the model and, therefore, dampened. On the other hand, real shocks do not have a direct impact on Ω_t that affects the share of vulnerable banks. Considering real shocks, P_v increases after a TFP shock. The effect is not extremely large in magnitude but is very persistent in time. The more the economy is active (with pressure on demand or an increase in production) the larger financial markets are, with banks supplying more loans and having higher profits. That increases the amount of funds banks trade one another and the overall connectivity of the system. As a result, the probability of being part of a vulnerable cluster increases. On the contrary, a consumer preference or a government spending shock decrease P_v, as they reduce the amount of final and interbank loans. Banks become less connected (z decreases)[86] and the number of banks being part of a vulnerable cluster is reduced.

[86]See Figures B.IV and B.V in Appendix C.1.

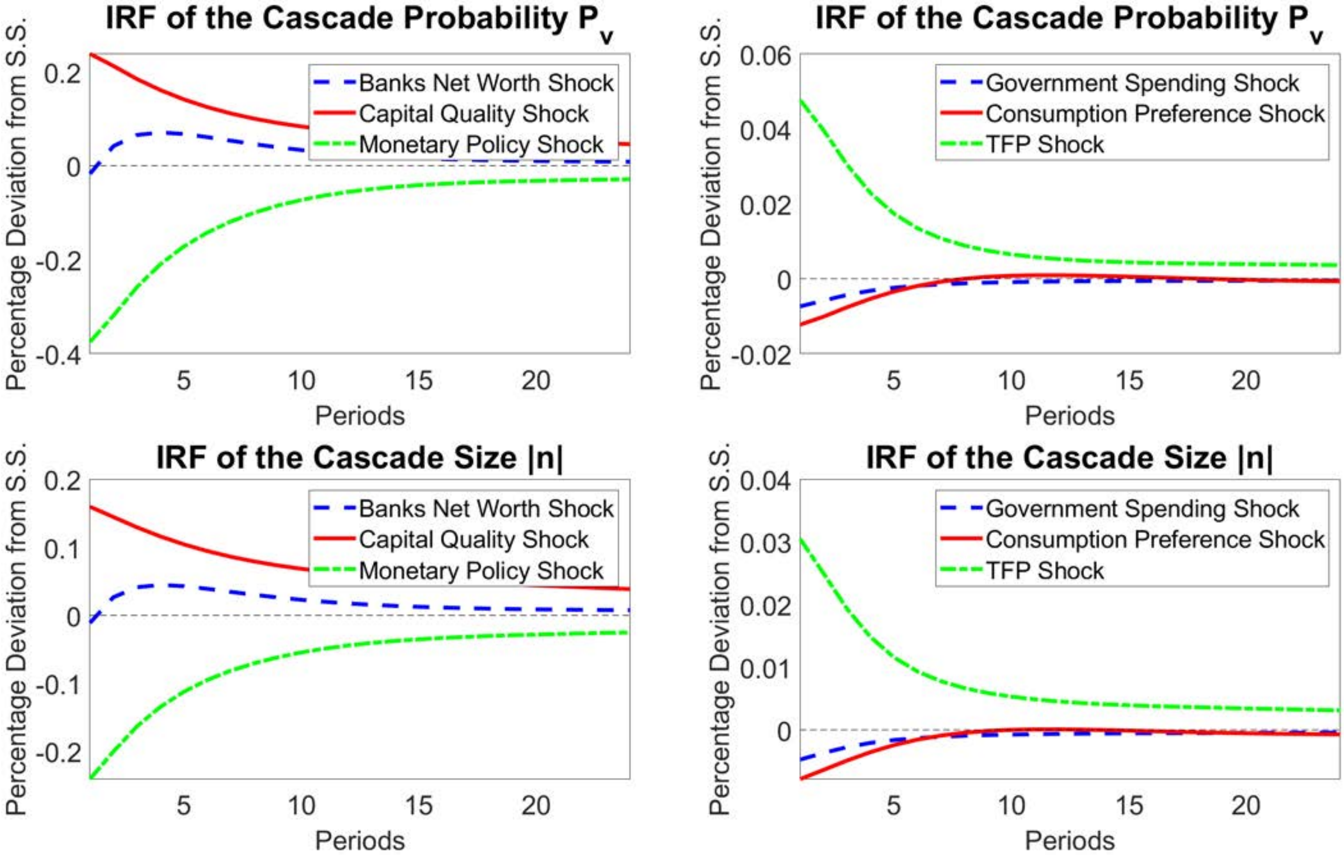

Figure 4.6: Cascade probability P_v and size $||n||$ responses to shocks.

A contractionary monetary policy surprise drives down significantly the probability of contagion -i.e. being part of a vulnerable cluster. As the policy rate increases, Ω increases, leading to, ceteris paribus, an increase in P_v. This effect, however, does not compensate the impact of the output contraction. As output falls banks finance less loans to firms and, consequently, demand less interbank credit. This is reflected on the interbank market, where banks are less connected; as financial institutions share less links, the cascade probability declines.

Capital quality and banks' net worth shocks, on the contrary, have a positive impact on P_v. The reasoning is similar to the previous case. Banks finance more loans to firms and, therefore, demand more funds on the interbank market. This last result is quite significant as highlights, from a theoretical point of view, the connections between an asset price boom and an increase of systemic fragility.

From the analysis of the response of P_v to shocks we can conclude that: i) the probability of being able to trigger a cascade -i.e. contagion- is affected not only by financial shocks but also by real ones, ii) real shock impact less on the cascade probability, as they influence only indirectly network variables, iii) real shocks, even if small in the initial magnitude tend to be long lasting, iv) policy makers face a trade off when it comes to monetary policy as stimulating output has the price of a more risky financial system, v) there is not a "divine coincidence" in this case, as an expansionary monetary policy has a cost in terms of financial stability, vi) government spending shocks have a positive effect on output with limited consequences on contagion.

It is not sufficient, however, to consider only the probability of a cascade to occur; economists and policy makers should also be concerned by its size ($\parallel n \parallel$ in the model). It grows, almost intuitively, in P_v and in the probability of having a vulnerable neighbor and decreases in the total size of the interbank market and in $G_0''(x)$. In the case of large financial markets, when z is large, banks are more distant and, therefore, clusters are more likely to be isolated.

The cascade size follows a path similar to the probability of contagion, only larger in magnitude. However, the dynamics involved are complex and highly non linear. The intuition is that $\parallel n \parallel$ depends positively on P_v, $G_0'(x)$ and $G_0''(x)$; these network measures are affected in the same way by shocks as the moments of $G(x)$ are increasing in the same set of variables. The only variable that moves in the opposite direction is z as the larger and less interconnected is the network the less likely a seed is able to grow.

To conclude, IRFs of the cascade size reinforce the conclusions drawn before. Real shocks have an effect on the stability of financial markets and policy makers face an additional trade off implementing monetary policies.

Figure B.III presents the IRFs of P_v and $||n||$ to the same shocks under the alternative assumption that banks trade their net worth on the interbank market. The IRFs of netwoek variables are broadly similar to the baseline model and the main conclusions hold when constructing differently the problem of banks. The only major difference is that the responses to financial shocks are amplified by about 30% while the impact of real shocks is broadly similar. Differences are due to lower sensitivity of bank's net worth to real variables. The interest rate on deposits, in fact, is pinned down by the Euler equation, which links the marginal utility of consumption to returns on savings. As such, they respond stronger (and faster) to changes in output than bank's net worth.

5 Conclusion

This paper delivers three main results. First, it provides a rigorous way to incorporate a network model into a standard general equilibrium framework. This methodology can be adapted to any model and translated to other sectors of the economy where diffusion processes and cascades dynamics matter (i.e. technological diffusion, information diffusion...).

Second, as network dynamics are highly complex, the characteristics of the starting node are not sufficient to understand the entire process. Its position in the network and the position of its first and second neighbors matters as well. This model takes into account that complexity when defining the network and incorporates the effect of the interactions between bank-specific and aggregate variables.

The paper additionally analyzes how contagion (in terms of likelihood and severity) evolves along the business cycle, integrating a complex network structure into a DSGE framework. Simulations show how systemic risk is influenced by financial and real shocks (with the latters having a smaller impact). Central banks face a trade-off when implementing monetary policy. Easing policies have a positive effect on output but only at the price of a greater instability on financial markets. This is due to the effect on banks' balance sheets of easing shocks. Simulations show also that monetary policy alone is not able to achieve financial stability without costs in terms of output stabilization.

Overall, the network structure and contagion risk are deeply affected by real and financial variables. In this complex framework there is the possibility that a marginal episode leads to consequence of systemic relevance. Future extensions of this work may be the integration into an open economy model of network dynamics, the inclusion of macroprudential policies and the estimation of the model itself. In particular this last exercise could lead to the identification of the key network parameter ($\widetilde{\gamma}$) without using restricted data.

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Appendix A Network Construction

Appendix A.1 Construction of $H_0(x)$ and $H_1(x)$

The average cluster sizeis defined as in Baumann & Stiller (2005). Assume that:

- the network contains no cycles, that is asymptotically true for large networks[87]
- for any edge between two nodes, i.e. a and b, the degree of b does not depend on a's neighbors or on the degree of a

Consider a general moment generating function $D_0(x)$ and k, the degree distribution, as a random variable. The sum of m realization of a random variable X is nothing else than:

$$S = X_1 + X_2 + \ldots + X_m$$

in terms of a generating function, the generating function of S is:

$$D_S(x) = D_{X_1}(x)\, D_{X_2}(x) \ldots D_{X_m}(x)$$

summing over the degree distribution of m randomly chosen variables (nodes in case of a graph), we have:

$$[D_0(x)]^m = \left[\sum_k p_k x^k\right]^m = \left[\sum_k p_k x^k\right]\left[\sum_j p_j x^j\right] = \sum_{jk} p_j p_k x^{j+k} \tag{A.1}$$

the coefficient of the power of x^n is the sum of the products $p_j p_k$ such that $j + k = n$ and gives the probability that the sum of the degrees of the two vertices is n.[88] Therefore, to compute the sum of the distribution of n randomly chosen vertices, extracted by a generating function S it is necessary to compute $[D_0(x)]^m$. Define the average degree of second neighbors of a vertex λ as the sum over the neighbors of the n first neighbors of λ, so as $[D_0(x)]^n$. Using equation (A.1), we can write $D_1(x)$, the average degree of the first neighbors of λ, as:

[87] See Newman et al. (2001).

[88] Intuitively, the sum over $p_j p_k$ gives the probability that any combination of two random extractions gives exactly n.

$$D_1(x) = \frac{\sum_k kp_k x^{k-1}}{\sum_j jpj} = \overbrace{\sum_k p_k \left[\underbrace{D_1(x)}_{\text{neighbors of a random vertex}} \right]^k}^{\text{probability of having k neighbors}} = D_0(D_1(x)) \tag{A.2}$$

Consider now the probability distribution $H_1(x)$ of the cluster size of a neighbor of a randomly chosen initial vertex λ. This is the probability that a cluster attach to one neighbor of a vertex a has exactly size s. If q_k is the probability of one of this neighbor to have k edges in addition to the one connected to the starting node, $H_1(x)$ must satisfy:

$$H_1(x) = xq_0 + xq_1 H_1(x) + xq_2 [H_1(x)]^2 + \dots$$

in other words, the distribution of the cluster size attached to a neighbor of an initial vertex is given by the average number of connection of the neighbor times the cluster connected to each of them. Using again equation (A.1) and realizing that q_k is just the coefficient on x^k of $D_1(x)$, $H_1(X)$ can be defined as:

$$H_1(x) = xD_1(H_1(x)) \tag{A.3}$$

Define now $H_0(x)$ as the size of the whole component, so the size of the component constituted by λ, its first neighbors and all the components (described by $H_1(x)$) attached to them. This component is nothing else than the sum of the components attached to all neighbors of λ. Therefore, using the previous reasoning, it is:

$$H_0(x) = xD_0(H_1(x)) \tag{A.4}$$

Consider now the diffusion model. $H_0(x)$ is the average vulnerable cluster size attached to a randomly chosen vulnerable vertex λ and $H_1(x)$ the average cluster size for a neighbor of $\lambda.H_1(x)$ must satisfy:

$$H_1(x) = x\left\{P_v^0 q_0 + \left[1 - P_v^0\right]\right\} + x\left\{P_v^1 q_1 H_1(x) + \left[1 - P_v^1\right]\right\} + x\left\{P_v^2 q_2 [H_1(x)]^2 + \left[1 - P_v^2\right]\right\} + \dots \tag{A.5}$$

with P_v^k the probability of a vertex with k edges being vulnerable. The previous equation states that the vulnerable cluster attached to a first neighbor of λ is proportional to the probability of this neighbor to be vulnerable given its degree distribution $\left(P_v^k q_k\right)$. In case the neighbor is not vulnerable, than the vulnerable cluster attached to it is just λ. Using the properties seen so far, (A.5) becomes:

$$H_1(x) = xG_1(H_1(x)) + \sum_k \left[1 - P_v^k\right] \tag{A.6}$$

P_v^k is equation (26) and therefore $\sum_k \left[1 - P_v^k\right]$ is the probability of a vertex to be non vulnerable. This is equation (30). Using the same argument of equation (A.4), equation (31) can be derived.

Appendix A.2 Large networks contain no cycles

This argument follows Gross & Yellen (2003). Define x the number of cycles in a random graph V with n vertices and p the edge probability. Given k vertices in the cycle, there are $\left(\begin{array}{c} n \\ k \end{array}\right) \frac{(k-1)!}{2}$ cycles of lenght k. The expected number of cycles is:

$$E(x) = \sum_{k=3}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{(k-1)!}{2} p^k \leq \sum_{k=3}^{n} (np)^3 = (np)^3 \frac{1-(np)^{n-2}}{1-np} \tag{A.7}$$

as $\lim_{n\to\infty} np = 0$, p is asymptotically less than $1/n$ thus the graph has no cycles.

Appendix A.3 Computing the average cluster size

Recall equations (30) and (31). Taking the first derivative of $H_1(x)$:

$$H_1'(x) = G_1(H_1(x)) + xG_1'(H_1(x))\, H_1'(x)$$

$$H_1'(x) = \frac{G_1(H_1(x))}{1 - xG_1'(H_1(x))} \tag{A.8}$$

with a similar process it si possible to define $H_0'(x)$ as:

$$H_0'(x) = G_0(H_1(x)) + xG_0'(H_1(x))\,H_1'(x)$$

plugging it into equation (A.8):

$$H_0'(x) = G_0(H_1(x)) + xG_0'(H_1(x))\,\frac{G_1(H_1(x))}{1 - xG_1'(H_1(x))} \tag{A.9}$$

recall now equation (27) and derive it to get $G_1'(x)$. The previous equation becomes:

$$H_0'(x) = G_0(H_1(x)) + \frac{x\left[G_0'(H_1(x))\right]^2}{z - G_0''(H_1(x))}$$

the average cluster size is $H_0'(x) \mid x = 1$, therefore:

$$H_0'(x) = \|n\| = P_v + \frac{z_v^2}{(z - G_0''(1))} \tag{A.10}$$

since $H_1(x)$ is a well behaving moment generating function with $H_1(1) = 1$.

Appendix A.4 Definition of ϕ

Define ϕ as the fraction of neighbors of a bank that can default without leading to the default of the bank itself.

$i = \frac{Int_t}{k}$ is the share of funds landed to each partner on the interbank market. As banks are all equal, each bank is indifferent between available borrowers. In order to reduce risk, therefore, it is optimal to differentiate.

Compute F, the maximum number of clients that may default without threatening the bank itself:

$$F = \frac{V}{i}$$

Recall the definition of ϕ:

$$\phi = \frac{F}{k}$$

use the definition of F to get:

$$\phi = \frac{V}{i}\frac{1}{k}$$

using the definition of i we get:

$$\phi = \frac{V}{k}\frac{k}{Int} = \frac{V}{Int}$$

Appendix A.5 Probabilities, Cascades and the Banking Network

A key question for policy makers is how to reduce the probability and size of contagion, minimizing the likelihood of a phase transition. In the terms of network theory that means that vulnerable nodes need to be insulated from the shock.[89] This result can be achieved with subsidies to banks' capital. As availability of funds and public support for these measures are limited, policy makers should identify which bank is more efficient to target with capital injections. Simulations show that it is generally better (for reasonable values of the immunized share of banks) to intervene on the higher part of the distribution, targeting the more connected banks. Preventing the default of a very central node, in fact, stops the cascade before it reaches more nodes and hence is more effective than targeting many (peripheral) small agents.[90] Results are reported in Figures A.I and A.III.

[89] See for example: Keeling & Eames (2005), Newman (2002), Pastor & Vespignani (2001), Keeling & Eames (2005), Gai & Kapadia (2010), Sterbenz et al. (2010) and Capponi & Chen (2015).

[90] Notice also that well connected nodes are more likely to be part of the cascade rather than peripheral nodes.

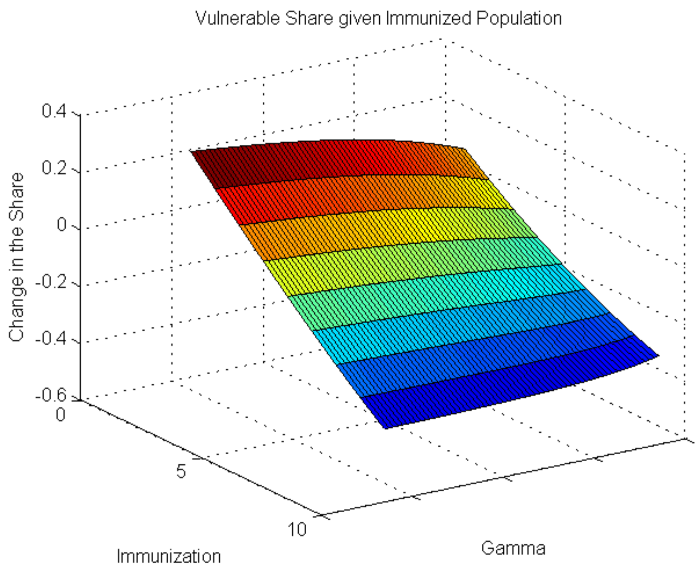


Figure A.I: Probability of a cascade, lower versus top.

Notes: the graph shows the difference in the share of banks that participate in a cascade (that equals the probability of a cascade) between the case in which the same share of banks is subsidies in the bottom or in the top of the distribution. A positive value means that P_v is larger in the case of subsidies to the least connected institutions.

Appendix A.6 Network analysis

To analyze how a change in the parameters of the network structure affects the cascade equations,the core set of equations (26)-(32) describing the network structure are evaluated for different values of their parameters: $\tilde{\gamma}$ and the average network degree z. These results show how cascade probabilities and sizes change if the network becomes more or less connected.

Equation (26) describes the probability that a defaulting bank triggers a cascade. Its main parameter is $\tilde{\gamma}$ that defines how concentrate is the network. Low values of $\tilde{\gamma}$ mean that the share of agents in the network with high degree is relatively larger, and vice versa. For the properties of power law distributions,[91] values of $\tilde{\gamma}$ used in this simulation fall in the interval $[2, 5]$. Empirical studies show that reasonable values for $\tilde{\gamma}$ lie in the interval $(2, 3)$ for existing interbank networks.[92]

The straight forward interpretation of the simulation is that the more a network is sparse, the more is likely that an agent belongs to a cluster. In sparse networks agents are divided between many small cluster. If an agent in the group defaults it is more likely that the default affects at

[91] See Aiello et al. (2001), Newman et al. (2001),Clauset et al. (2009).

[92] See for example Boss et al. (2004*a*), Boss et al. (2004*b*), Caldarelli et al. (2006) and Soramaki (2007).

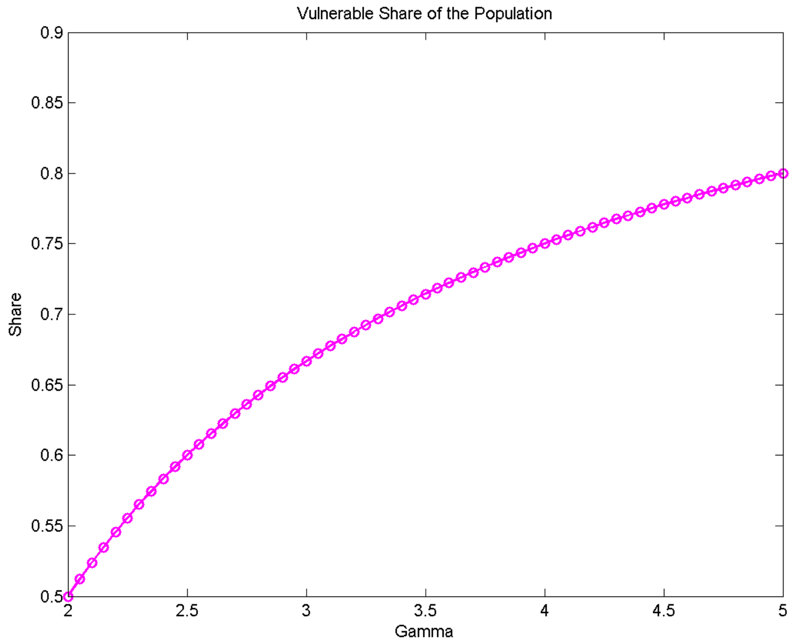


Figure A.II: Probability of starting a cascade after an initial default, from equation (26).

least another member of the cluster. However, figure A.II should not be misleading. Equation (26) describes the probability that a randomly chosen agent belongs to a vulnerable cluster, however it tells nothing about the cluster size.

With more sparse networks clusters are also smaller. Therefore the correct interpretation of figure A.II is the following: as the network becomes more sparse, there is an higher probability that a randomly chosen vertex belongs to a cluster, however the dimension of the cluster decreases with $\widetilde{\gamma}$. As the probability of a cascade increases with $\widetilde{\gamma}$ the consequences of contagion become smaller, as nodes are more isolated and an initial seed can infect fewer neighbors.

To verify this intuition, equation (32) is evaluated at different values of $\widetilde{\gamma}$ and z. Before moving on, it is probably worth to examine the cascade condition more in details. It is influenced by all network variables: P_v, z_v (the average degree of vulnerable vertices) and z (the average degree of the whole network). These variables are combined in equations (31) and (30) to define $\parallel n \parallel$. They, however, push in opposite directions as, for example, equation (27) clearly shows how z_v decreases as $\widetilde{\gamma}$ increases. The higher is $\widetilde{\gamma}$ the fewer neighbors each vulnerable vertex has. On the contrary, $\parallel n \parallel$ depends positively on $G_0''(x)$. This term has not a clear economic interpretation, but captures the second moment of the distribution of vulnerable vertices.[93] Notice that it depends positively

[93]See Newman et al. (2001) and Watts (2002).

on $\widetilde{\gamma}$.

Finally $\parallel n \parallel$ depends negatively on the whole network average connectivity z. Intuitively, the larger is z the more agents are interconnected; as connectivity rises each agent gets a smaller share (negligible in the limit) of default costs from the initial seed.[94]

z-$G_0''(x)$ defines the so called "phase transition equation".[95] When $z = G_0''(x)$ equation (33) has a singularity as $\parallel n \parallel \rightarrow \infty$.[96] If the phase transition condition is met, the initial seed grows up to the point that it covers the entire graph, independently by its initial size. In the terms of the interbank network that means that the default of a single institution leads to the collapse of the entire market structure. The conditions for a phase transition root deep into the network structure, and are not easily pinned down. What is clear, however, is that it occurs only at specific combinations of variables. Notice that the shift to a phase transition is sudden and driven by an apparently small change of the variables.

Figure A.III shows $\parallel n \parallel$ evaluated at different values of z and $\widetilde{\gamma}$. The spikes in the graph are regions close to a phase transition. As $z - G_0''(x) \rightarrow 0$, $\parallel n \parallel \rightarrow \infty$. The closer the values of z and $\widetilde{\gamma}$ are to the actual combination that generates a phase transition, the higher is the share of nodes participating in the cascade and the higher is the spike. In the limit, for that specific combination of parameters such that $z = G_0''$, $\parallel n \parallel$ is infinite. Phase transitions are concentrated around low values of $\widetilde{\gamma}$.

To build on what we discussed before, the cascade size decreases with $\widetilde{\gamma}$ for the intuitive reason that on more sparse networks vulnerable seeds are less connected.[97] The opposite reasoning can explain why it decreases with z. As the average connectivity grows, on one hand it is more likely that vulnerable nodes are connected to non vulnerable ones on the other each node receives a smaller share of the shock. These two effects reinforce each other.

[94] A similar argument is developed in Acemoglu et al. (2015).

[95] Equation (33).

[96] See Stanley (1971), Stauffer & Ahrony (1991), Callaway et al. (2000), Watts (2002), Falconer (2003), Alentorn et al. (2007), Falconer (2003), Mandelbrot (2004), Krantz (2005), Alentorn et al. (2007).

[97] Notice that this means that the effect of $\widetilde{\gamma}$ on P_v prevails on that on G_0''.

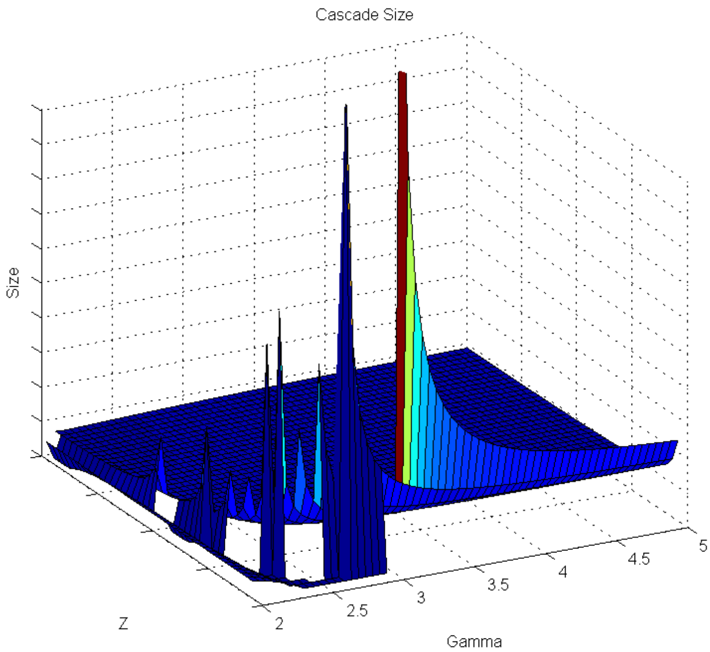


Figure A.III: Average cascade size, equation (32).

Notes: simulations for different values of z and $\widetilde{\gamma}$ of the size of contagion. The spikes in the graphs are associated to those regions where $z - G_0''(x) \rightarrow 0$. In those areas the system is getting closer to a phase transition (as $\parallel n \parallel \rightarrow \infty$).

Appendix B An alternative configuration of the interbank network

This appendix constructs the network under the alternative assumption that banks trade net worth, rather than a fraction of deposits, on the interbank market and that deposits are used in the creation of final loans to firms. This is more in line with the baseline assumption of Acemoglu et al. (2015) and provides an alternative framework to justify the presence of an active interbank market in the model. The robustness exercise delivers similar results to the baseline model, with the main difference that network variables respond more to financial shocks (by about 30%) and to real shocks. The difference is due to the lower sensitivity of net worth to real factors, as returns on deposits are pinned down by the Euler equations, that links consumption to savings. Real shocks, however, continue to generate reactions significantly smaller than financial shocks.

Appendix B.1 The problem of banks

The wholesale branch trades all available net worth on the interbank market. It enters the interbank market for the same reason discussed in the baseline model: investment opportunities and liquidity do not arise simultaneously. This branch acquires interbank funds to be transferred to the retail branch and invests liquidity (when available) financing other banks. The wholesale branch maximizes:

$$E_t(V_t^{WB}) = \sum_{i=0}^{\infty} \left(1-\theta^B\right)(\theta^B)^i \Lambda^C_{t,t+1+i}\left[Int_t^+ R^I_{t+1} - N_t R_{t+1} - Int_t^- R^I_{t+1}\right] \tag{B.11}$$

subject to the resource constraint $Int_t^+ \leq N_t$ and the demand of interbank loans by the retail branch $Int_t^+ = B_t - D_t$. R^I_{t+1} is the interbank rate and R_{t+1} the opportunity cost of unused capital. As there are no credit frictions on the wholesale market, the interbank market operates in perfect competitions; rates are set at the marginal cost and banks supply all funds available. First order conditions are:[98]

$$R^I_t = R_t \tag{B.12}$$

$$Int_t^+ = Int_t^- = N_t \tag{B.13}$$

[98]Notice that these conditions are equivalent to Gerali et al. (2010) in the *absence of frictions*.

The retail branch uses net worth and deposits to finance final loans. It maximizes the present discounted value of the branch subject ot the resurce constraint and the incentive compatibility constraint $(E(V_t) \geq \Theta B_t)$.

$$E_t\left(V_t^{RB}\right) = \sum_{i=0}^{\infty}\left(1-\theta^B\right)\left(\theta^B\right)^i \Lambda_{t,t+1+i}^C [R_{t+1+i}^K B_{t+i} - R_{t+1+i}^D D_{t+i} - R_{t+1+i}^I Int_{t+i}^-] \tag{B.14}$$

The optimal solution is:

$$E(V_t) = k_t^1 N_t + k_t^2 B_t \tag{B.15}$$

with k_t^1 and k_t^2 defined as:

$$k_t^1 = E_t\left[\left(1-\theta^B\right) + \Lambda_{t,t+1}^C \theta^B \frac{Int_{t+1}^-}{Int_t^-} k_{t+1}^1\right] \tag{B.16}$$

$$k_t^2 = E_t\left[\left(1-\theta^B\right)\Lambda_{t,t+1}^C \theta^B \left(R_{t+1}^K - R_{t+1}^D\right) + \Lambda_{t,t+1}^C \theta^B \frac{B_{t+1}}{B_t} k_{t+1}^2\right] \tag{B.17}$$

The loan supply is

$$B_t = \phi_t Int_t^- \tag{B.18}$$

$\phi_t^L = \frac{k_t^2}{\Theta - k_t^1}$ is the ratio between deposits and interbank funds.

Appendix B.2 Static and dynamic simulations

Similar to 4.1 Figures B.I and B.II present the static simulations of key network variables to changes in the leverage of banks and the spread between assets and risk-free securities. Figure B.III presents the IRFs of the P_v and $||n||$ to real and financial shocks. Results are in line with the main conclusions of the paper. The differences with respect to the baseline model are discussed in the main text.

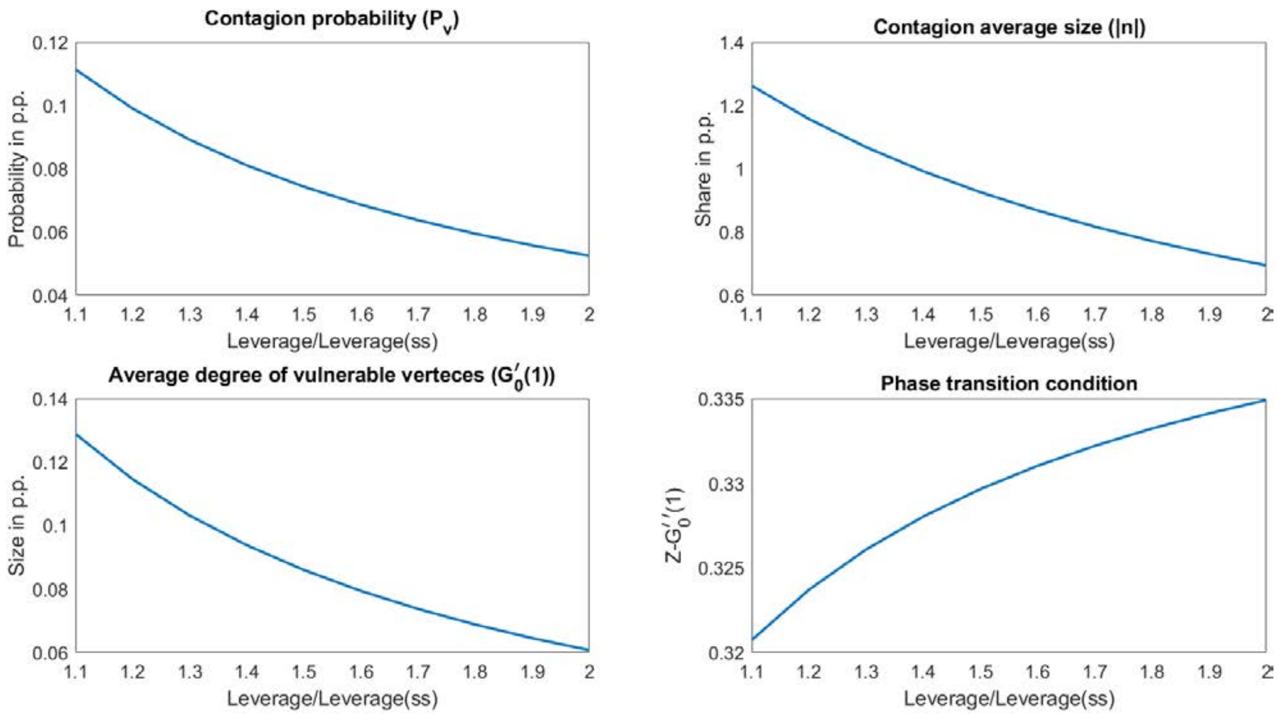


Figure B.I: Steady state evolution of key network variables changing the equilibrium leverage of financial intermediaries when banks trade net worth rather than a share of deposits on the interbank market.

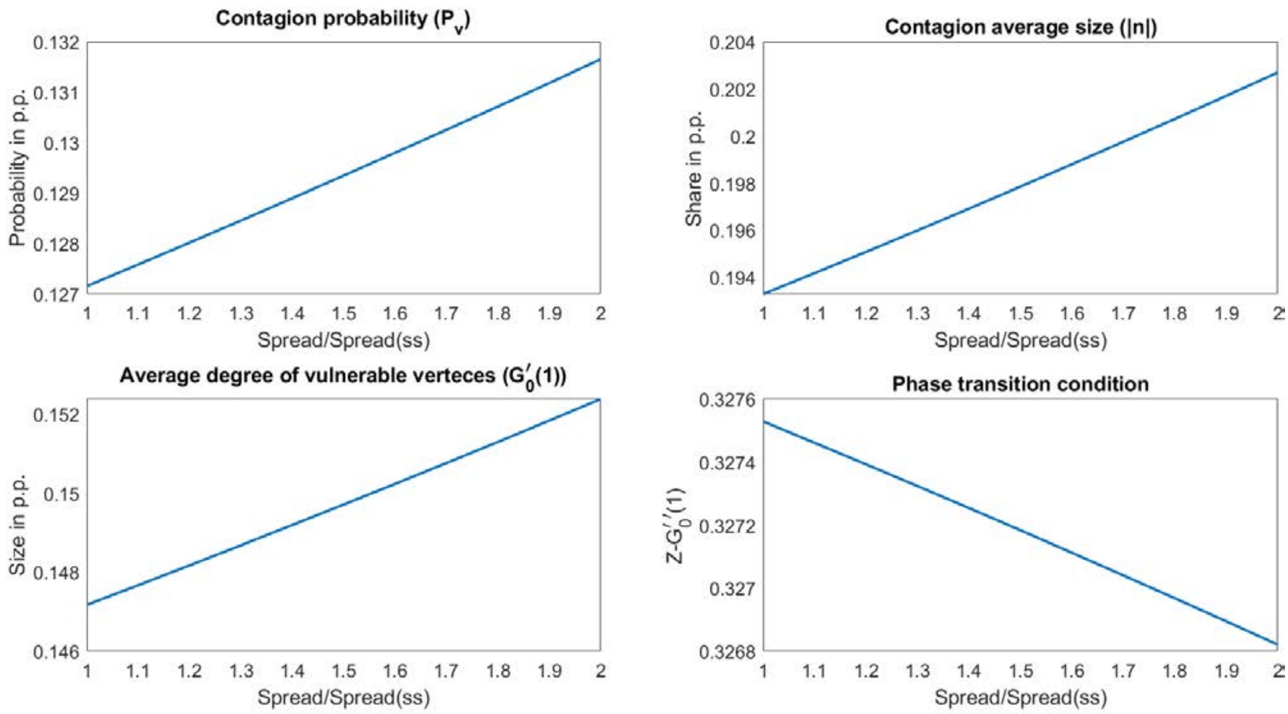


Figure B.II: Steady state evolution of key network variables changing the equilibrium spread $R^K - R$ when banks trade net worth rather than a share of deposits on the interbank market.

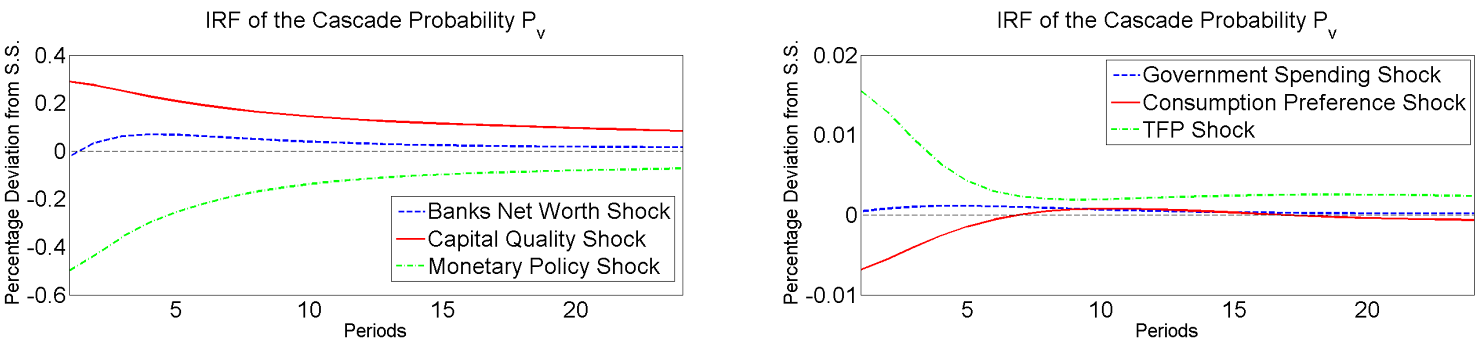
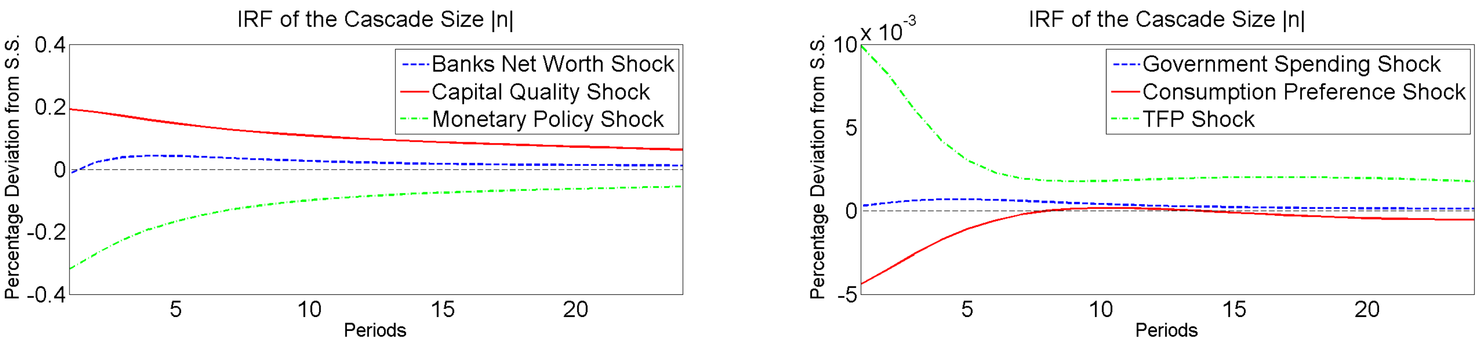


Figure B.III: Cascade probability P_v response to shocks when banks trade their net worth on the interbank market.



dyn$_{d}eposits - cont.Cascadesize\| n \|$ response to shocks when banks trade their net worth on the interbank market.

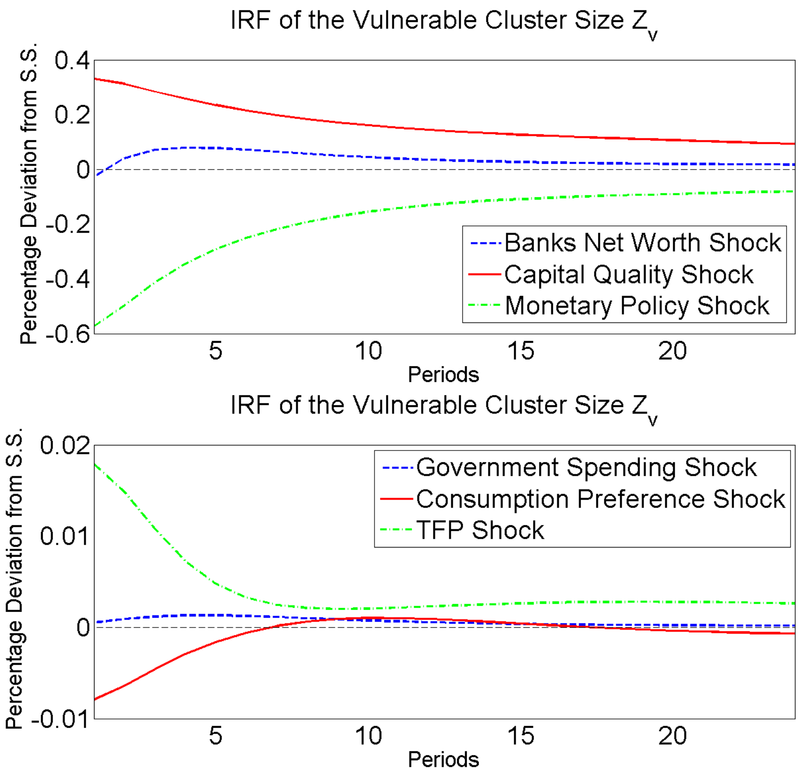


Figure B.IV: Average vulnerable cluster size (z_v) response to real and financial shocks when banks trade their net worth on the interbank market.

Appendix B.2.1 IRFs of the Average Cluster Size Z

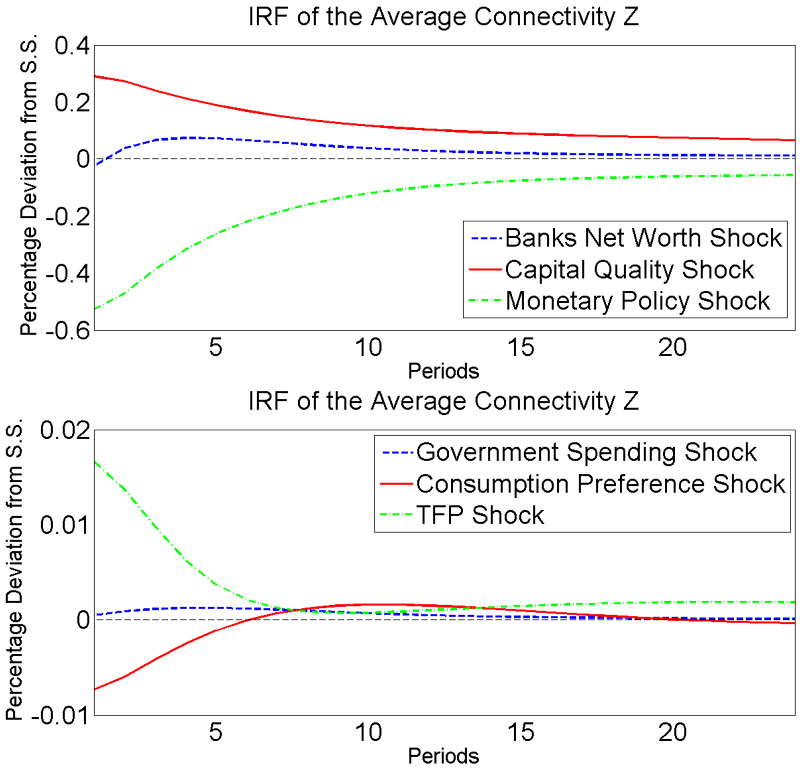
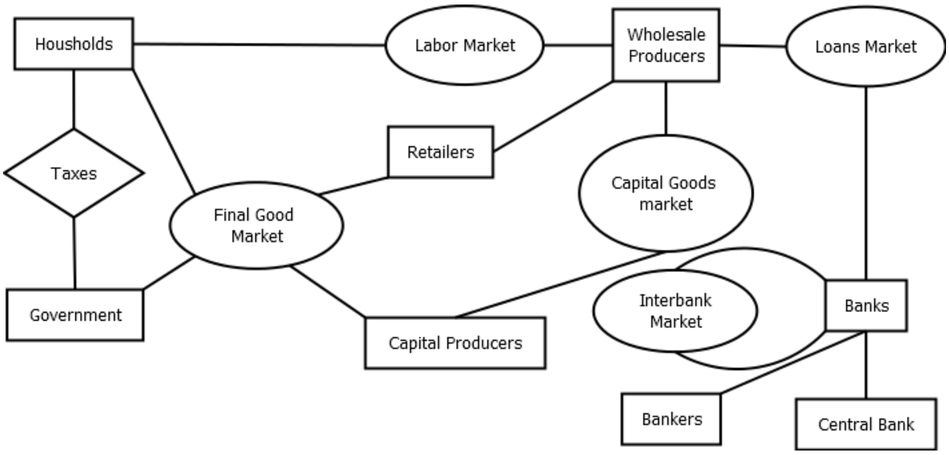


Figure B.V: Average cluster size (z) response to real and financial shocks when banks trade their net worth on the interbank market.

Appendix C Figures and Tables

Appendix C.1 Figures

Appendix C.1.1 The model



Structure of the model.

Appendix C.1.2 IRFs of Output

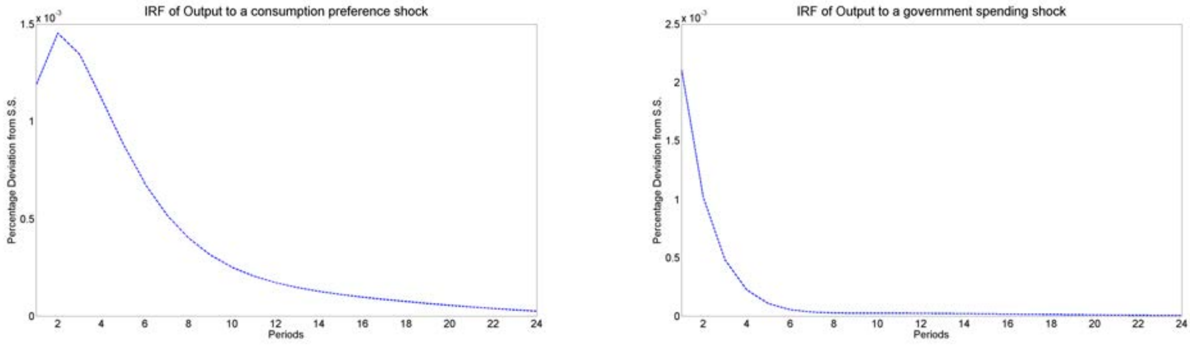


Figure C.I: Output response to consumer preference and government spending shocks.

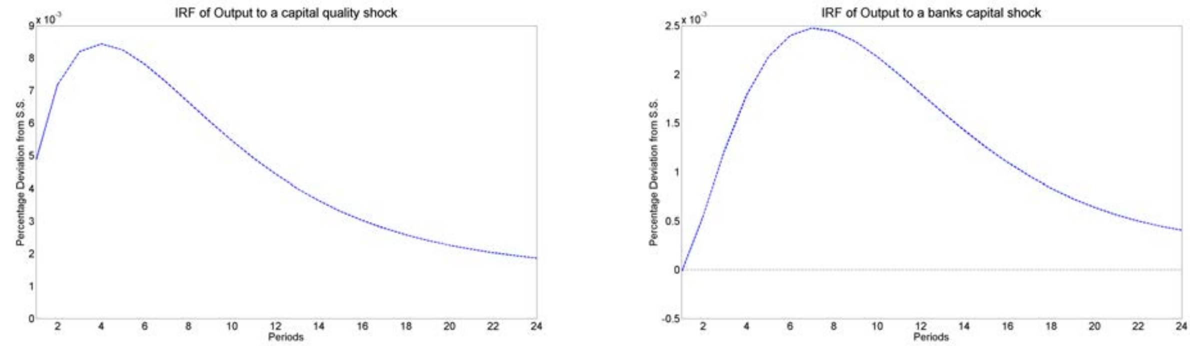


Figure C.II: Output response to capital quality and banks' net worth.

Appendix C.1.3 IRFs of Consumption

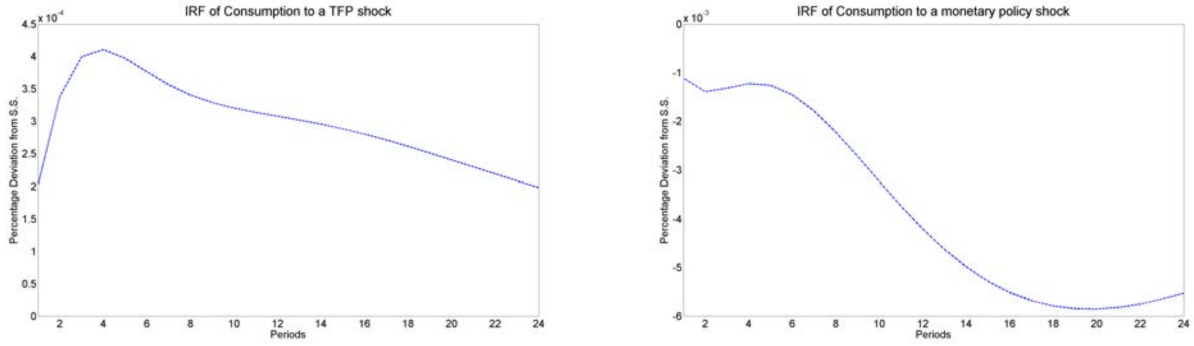


Figure C.III: Consumption response to TFP and monetary policy shocks.

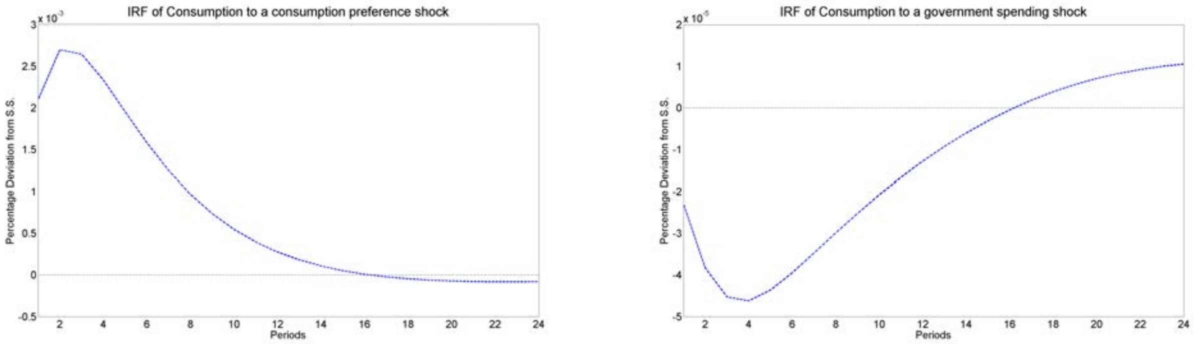


Figure C.IV: Consumption response to consumer preference and government spending shocks.

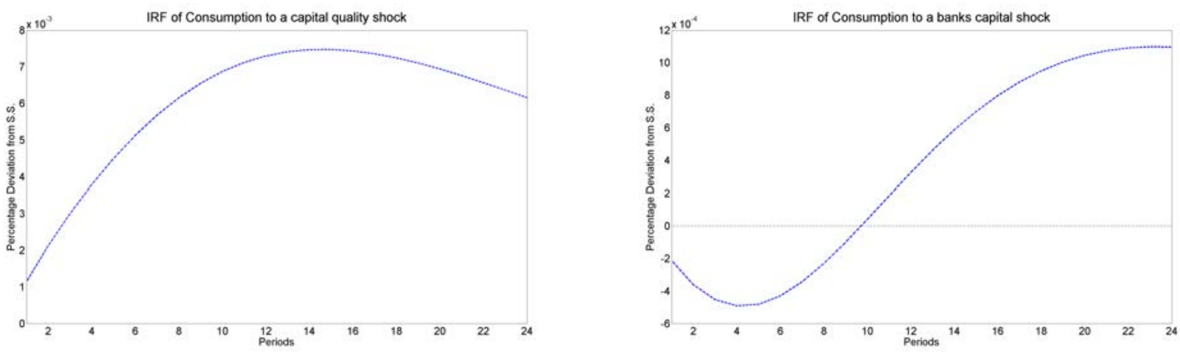


Figure C.V: Consumption response to capital quality and banks' net worth.

Appendix C.1.4 IRFs of Inflation

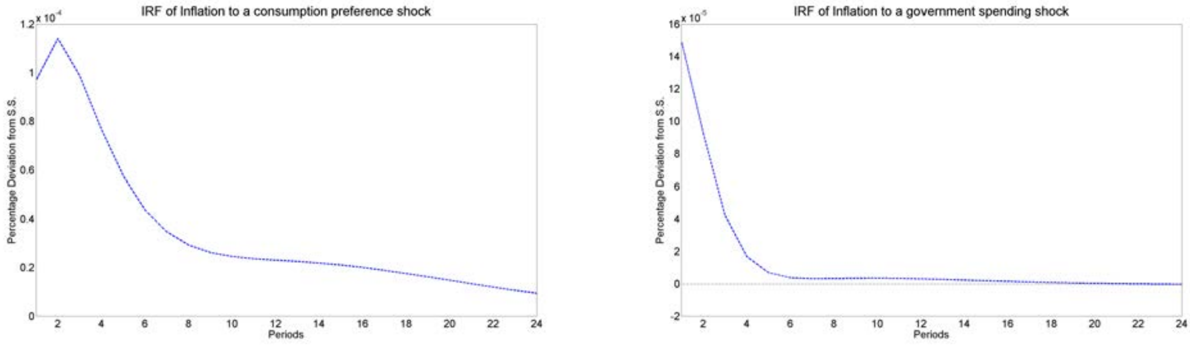


Figure C.VI: Inflation response to consumer preference and government spending shocks.

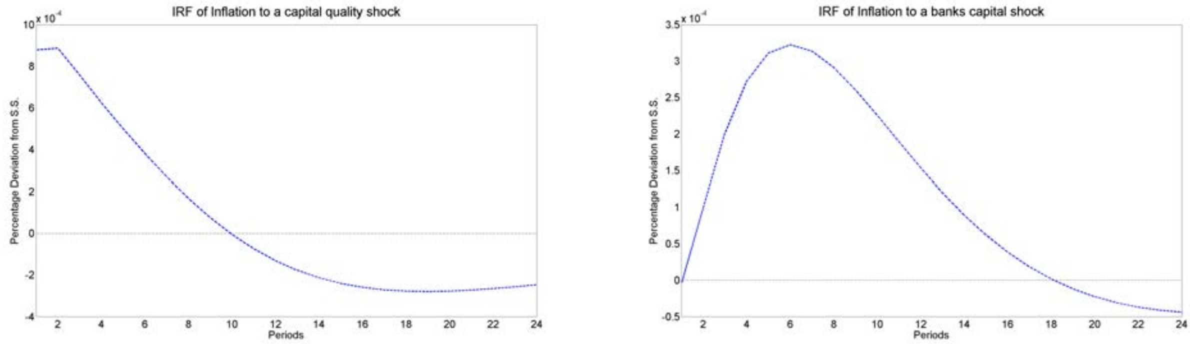


Figure C.VII: Inflation response to capital quality and banks' net worth.

Appendix C.1.5 IRFs of Total Loans

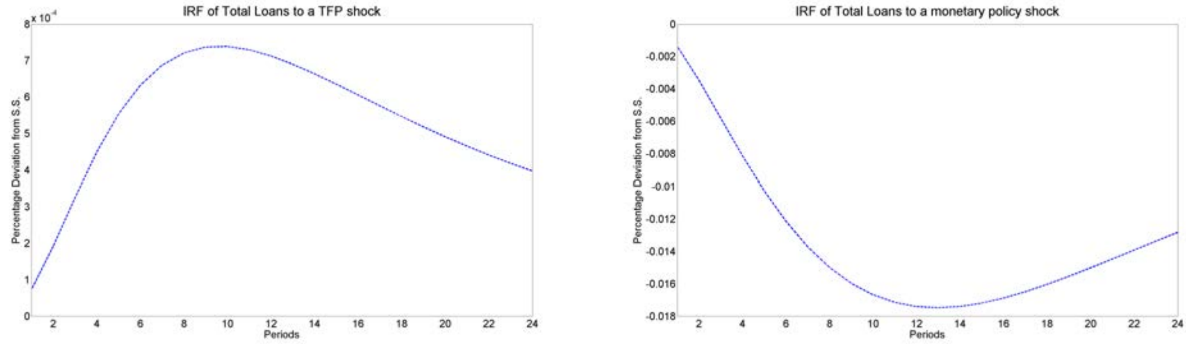


Figure C.VIII: Total loans response to TFP and monetary policy shocks.

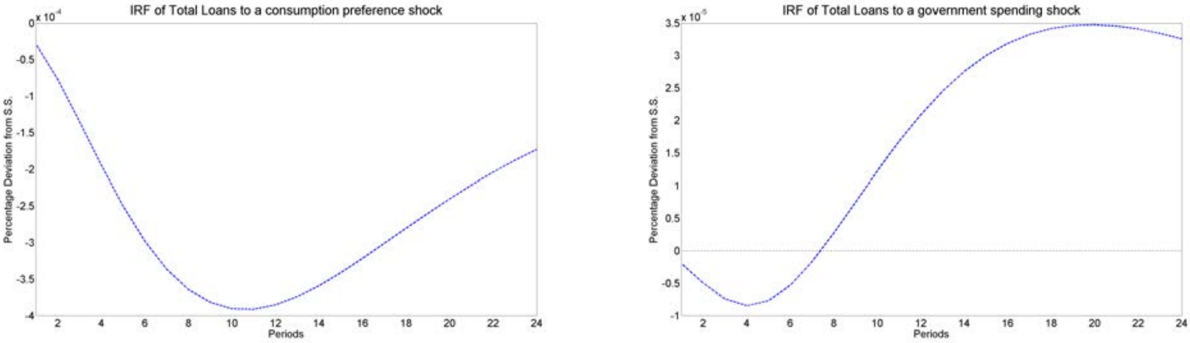


Figure C.IX: Total loans response to consumer preferences and government spending shocks.

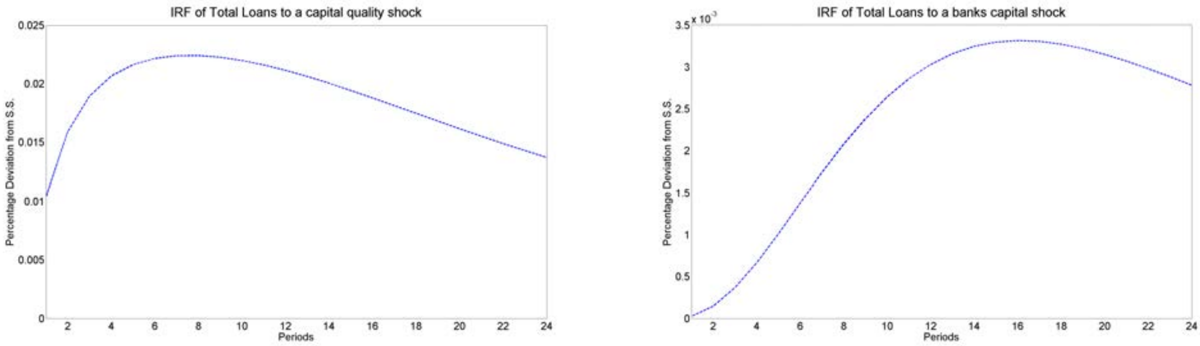
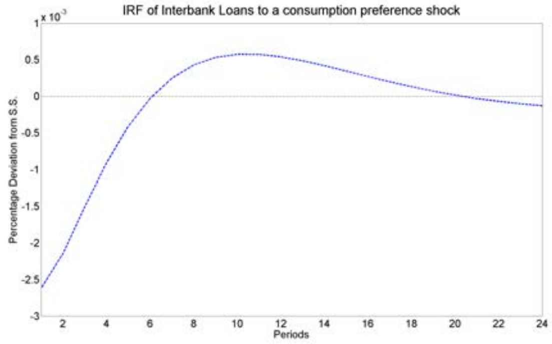


Figure C.X: Total loans response to capital quality and banks' net worth.

Appendix C.1.6 IRFs of Interbank Loans



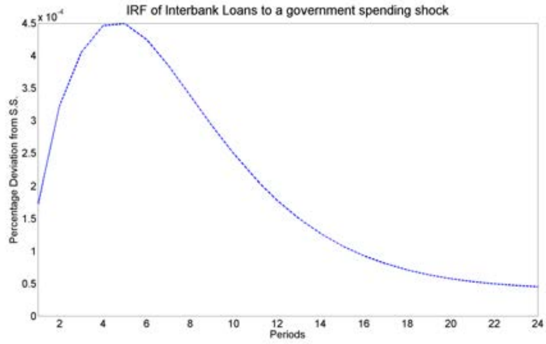
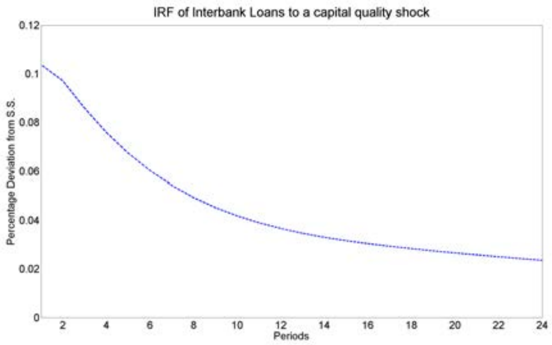


Figure C.XI: Interbank loans response to consumer preferences and government spending shocks.

Figure B9. Interbank loans response to consumer preferences



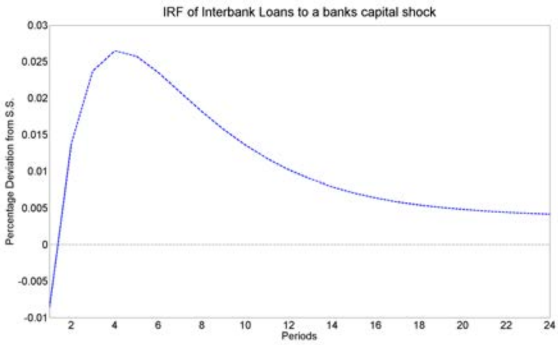


Figure C.XII: Interbank loans response to capital quality and banks' net worth.

Appendix C.1.7 IRFs of the Policy Rate

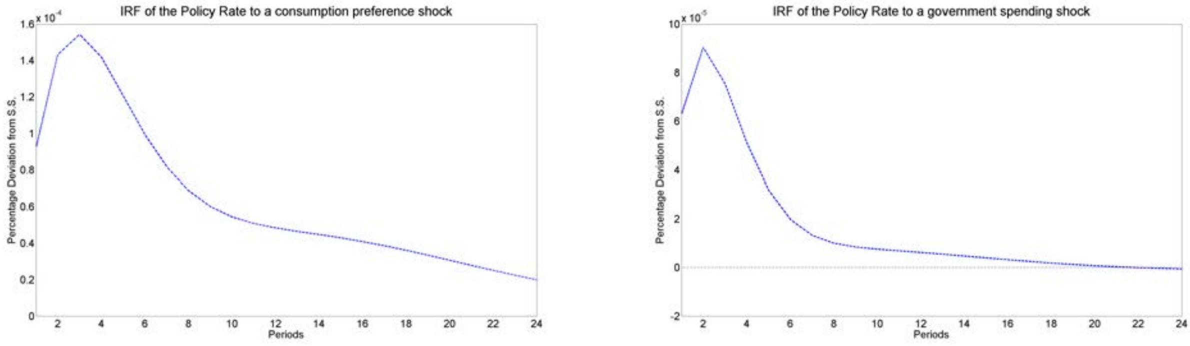


Figure C.XIII: Policy rate response to consumer preferences and government spending shocks.

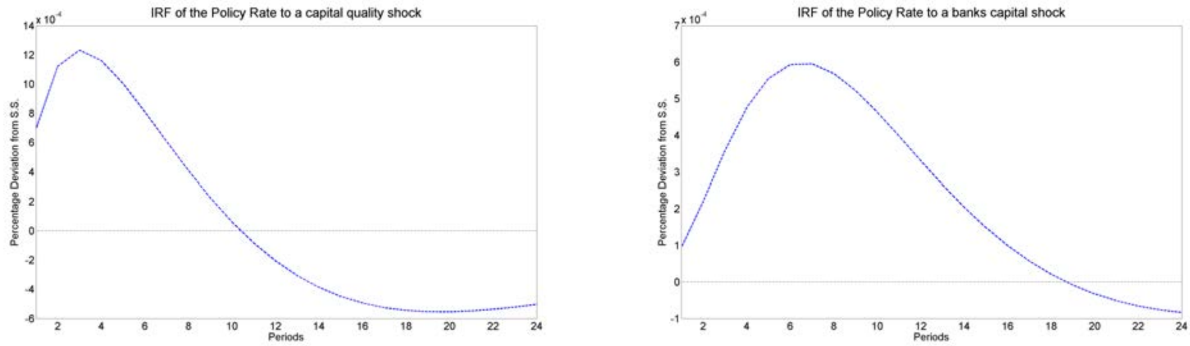
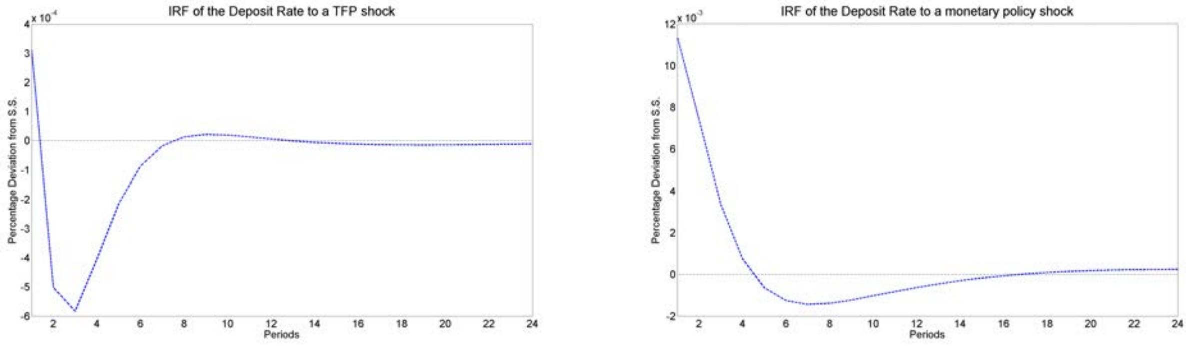


Figure C.XIV: Policy rate response to capital quality and banks' net worth.

Appendix C.1.8 IRFs of the Deposit Rate



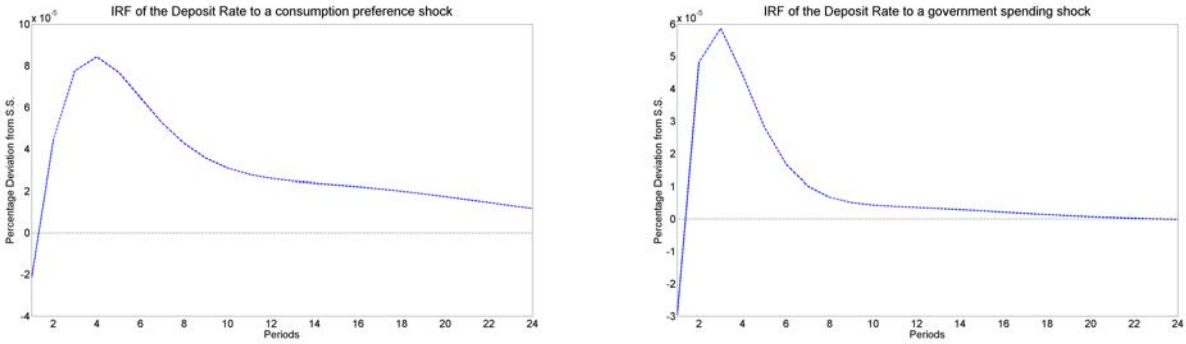


Figure C.XV: Deposit rate response to consumer preferences and government spending shocks.

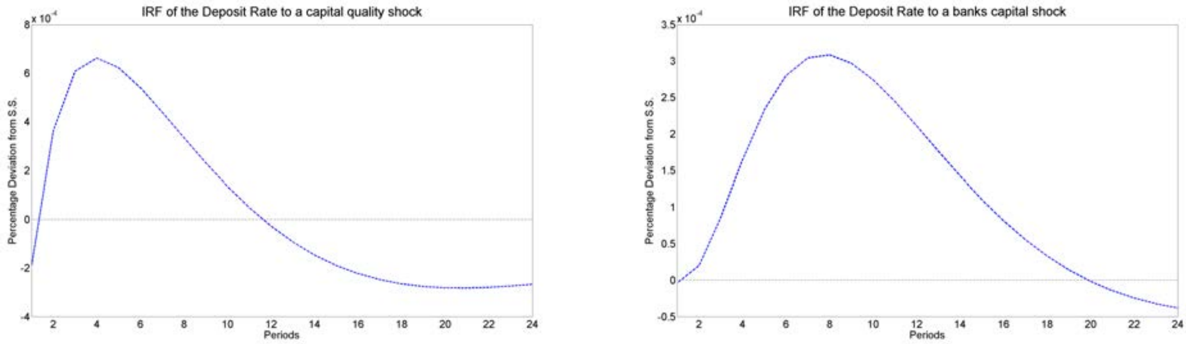


Figure C.XVI: Deposit rate response to capital quality and banks' net worth.

Appendix C.1.9 IRFs of the Average Vulnerable Cluster Size Z_V

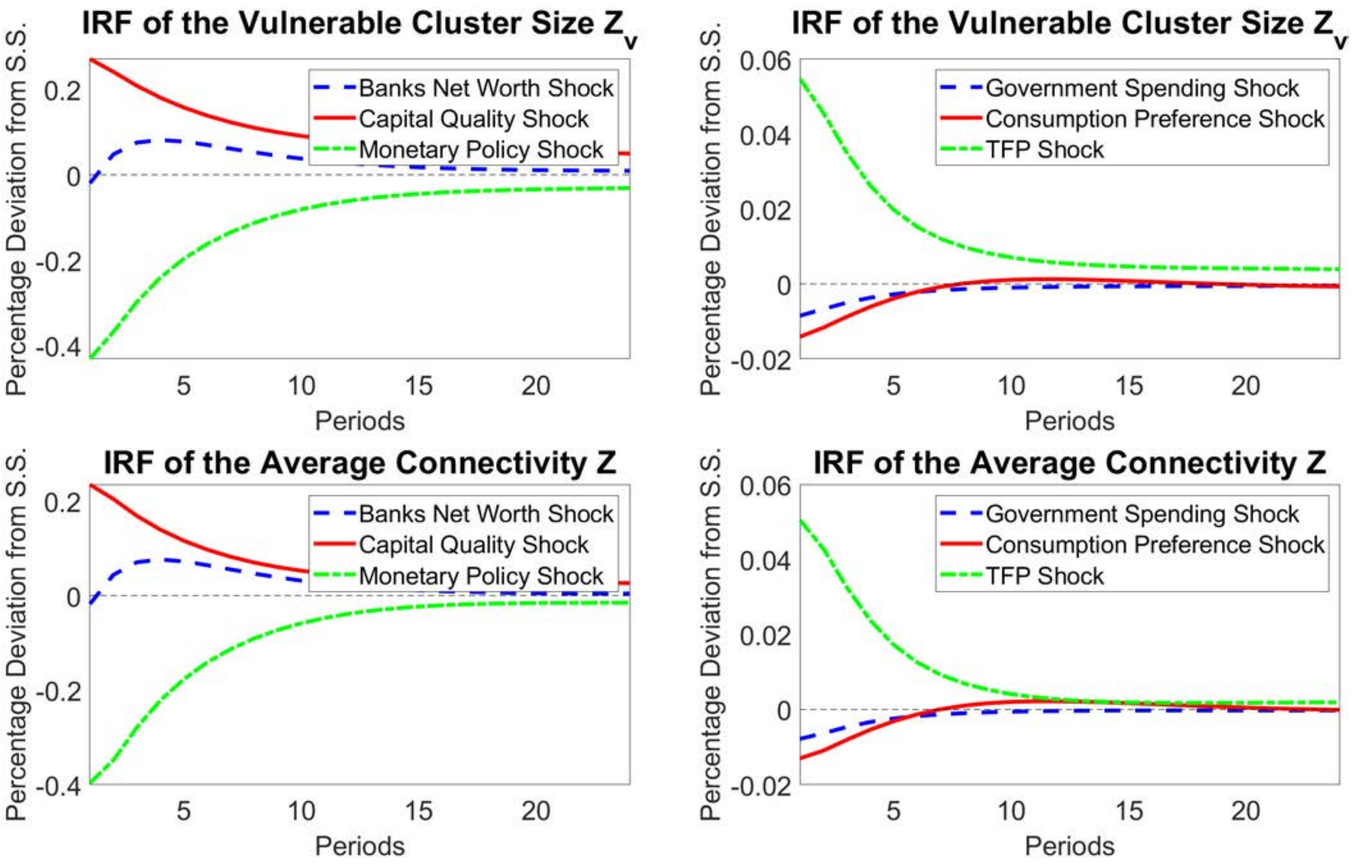


Figure C.XVII: Average cluster size (Z_v) and average connectivity ($||Z||$) responses to shocks.

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