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Adriana Grasso, Filippo Natoli

Consumption volatility risk and the inversion of the yield curve

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Abstract

We propose a consumption-based model that allows for an inverted term structure of real and nominal risk-free rates. In our framework the agent is subject to time-varying macroeconomic risk and interest rates at all maturities depend on her risk perception which shape saving propensities over time. In bad times, when risk is perceived to be higher in the short- than the long-term, the agent would prefer to hedge against low realizations of consumption in the near future by investing in long-term securities. This determines, in equilibrium, the inversion of the yield curve. Pricing time-varying consumption volatility risk is essential for obtaining the inversion of the real curve and allows to price the average level and slope of the nominal one.

JEL classification: G12.

Keywords: real rates; uncertainty; habits; inverted yield curve; volatility risk.

Non Technical Summary

In this paper we explain how uncertainty over the economic outlook has effects on the term structure of interest rates, i.e. on interest rates on government bonds at different maturities, through the consumption and saving choices of agents. In particular, we provide one explanation for the inversion of the term structure, that happens when interest rates on short-term bonds are above those on long-term ones. We claim that the variation in uncertainty of economic agents is one of the drivers of the inversion and we provide a theoretical model to explain the main transmission channels. As far as we know, there is no other paper providing the theoretical background to this phenomenon for the driver we have in mind.

In normal times, when the outlook of an economy is expected to improve over time, interest rates on its debt are increasing with the maturity (i.e. the term structure is upward-sloping): investors require a premium to buy longer-term securities because they renounce to the possible extra-returns obtained by buying short-term bonds and to roll their positions over at maturity. However, taking the United States as a reference economy and looking at the time series of interest rates, the inversion of the term structure cannot be considered a rare event.

Interest rates on government bonds reflect the combination of many factors, i.e. growth and inflation expectations, as well as the risk perceived around those expectations. While variations in the stance of monetary policy inducing changes in short-term inflation expectations are usually considered as the only responsible of the inversion, we show that another driver is also at place and should be considered by researchers and policymakers. Indeed, the term structure of interest rates on government inflation-protected securities has also inverted in the last 10 years, suggesting a role for the real component. Moreover, the empirical evidence shows that variations in market volatility are correlated to changes in the slope of the term structure of interest rates: in particular, inversions in the term structure of interest rates and in the term structure of market volatility (a proxy for short- and long-term economic uncertainty) have been synchronized in the past years.

We describe the effect of uncertainty on interest rates in a consumption-based asset pricing model. In our model, a representative agent has utility over one consumption good that comes exogenously in the economy, and she is uncertain about the quantity of consumption good that will be available in the future. In each period, she observes the

quantity of received consumption good and she updates her risk perception over future consumption, i.e. the risk that consumption good tomorrow will be low. Interest rates at all maturities reflect, in equilibrium, the consumption-saving desires of the consumer. Intuitively, when risk is perceived to be higher in the long- than the short-run, the agent would be more willing to invest in short- rather than long-term securities: in equilibrium, the yield curve is upward sloping. Vice versa, when risk is perceived to be higher in the short-term, the agent would prefer to hedge against low realizations of consumption in the near future by demanding long- instead of short-term bonds: this entails the inversion of the equilibrium real yield curve.

After having explained the basic mechanisms, we extend our model to include inflation expectations. The analysis of the term structure is important from a policy perspective: for example, it gives insights on the transmission of monetary policy and it provides signals on the business cycle dynamics, with some studies documenting that the inversion of the term structure is a predictor for future recessions.

1 Introduction

The inversion of the term structures of interest rates, which happens when short-term yields are above long-term ones, is an occasional, yet not rare event. Postwar data on US Government bond yields show that while the term spread - i.e., the difference between long- and short-term yields - has been positive on average, several episodes of inversion have also been documented (Figure 1.1).

Nominal yields reflect a real as well as an inflation expectations component. While the latter is known to play a role in shaping the level and slope of the yield curve, less evidence is available on the role of the real component. Stylized facts point to the role of real factors and, in particular, real macroeconomic risk in affecting the term structure of interest rates: first, data on US TIPS (i.e., inflation-protected securities) from Gurkaynak et al. (2007, 2010) suggest that real yields fluctuate substantially over time and that the real term structure of interest rates, as well as the nominal one, became inverted in a few occasions during the last ten years (see Figure 1.2); second, the slope of option-implied volatilities in the stock market (proxying long- vs. short-term uncertainty) is positively correlated with that of the TIPS yield curve, and the two have experienced synchronized inversions (see Figure 1.3).[1]

This paper investigates the role of macroeconomic risk in shaping the term structure of interest rates. For this purpose, we propose a parsimonious consumption-based model of the term structure that allows for the inversion of its real component. In our framework the representative agent is subject to time-varying macroeconomic risk and interest rates at all maturities depend on her risk perception which shape saving propensities over time. Intuitively, when risk is perceived to be higher in the long- than the short-run, the agent would be more willing to invest in short- rather than long-term securities: in equilibrium, the yield curve is upward sloping. Vice versa, when risk is perceived to be higher in the short-term, the agent would prefer to hedge against low realizations of consumption in the near future by demanding long- instead of short-term bonds: this entails the inversion of the equilibrium real yield curve.

Our model builds on the classic endowment economy frameworks of Campbell and Cochrane (1999) and Wachter (2006). In these models, a representative agent has consumption preferences with respect to a habit level, and variations in the surplus over habits drive both the desire to smooth consumption over time and a precautionary mo-

[1] Three out of four episodes of inversion since 2008 are concurrent with inversions in the volatility term structure; the only inversion of the yield curve that is not matched by that of the volatility term structure is the one of 2011-2012, which happened during the implementation of the *Operation Twist* by the Federal Reserve.

tive that depends on changes in risk aversion. By inducing opposite consumption-saving desires, these two forces have opposite effects on the implied equilibrium risk-free rate, and, potentially, on the slope of the real term structure. Campbell and Cochrane (1999) offset them to produce a constant risk-free rate, while Wachter (2006) makes the consumption smoothing motive always prevail so that, in equilibrium, the yield curve is always upward-sloping.

In the two papers mentioned above, the authors assume an exogenous, log-normal process for consumption growth. With respect to those frameworks, our model features time-varying volatility of consumption growth and imperfect information. Consumption growth is a Markov switching process in which volatility varies between two regimes; agents do not observe the volatility state but infer it from the available draws of consumption. In equilibrium, real interest rates depend not only on the level of consumption with respect to habits, but also on expected volatility in the next period relative to the long-run average. Indeed, expectations of high consumption growth volatility in the short-run causes the precautionary motive to prevail: in times of low consumption relative to habits, this entails a prevailing desire for precautionary savings, while in times of high consumption, a prevailing desire of borrowing. In both cases, the equilibrium real yield curve is inverted.

Following Wachter (2006), nominal yields are then constructed by adding an exogenous inflation process to the aforementioned framework. With respect to the real term structure, the nominal one depends also on nominal factors (i.e., short-term inflation expectations and inflation volatility) and on the correlation between consumption growth and inflation. Assuming a negative correlation between the two, inflation volatility adds to consumption growth volatility as a second source of risk for the agent. The cumulated perceived risk matters for the slope of the implied equilibrium nominal yield curve in the same way as consumption volatility risk matters for the real curve.

Our model is mainly inspired by three studies. The key feature of consumption growth volatility being unobservable and time-varying is taken from Boguth and Kuehn (2013), who explore the connection between macroeconomic uncertainty and asset prices, finding that consumption growth volatility predicts returns for risk-exposed firms; the emphasis on long- vs. short-run risk is in the spirit of Bansal and Yaron (2004), who propose plausible solutions to asset pricing puzzles based on a persistent component in expected growth and on fluctuating uncertainty; the role of macroeconomic shocks on the slope of the yield curve is in line with Kurmann and Otrok (2013), who find that news about future total factor productivity (TFP) is the main factor behind the inversion of the curve and suggest that those shocks are linked to consumption growth volatility. Finally, intuitions about the connection between interest rates and macroeconomic risk are also in Breeden et al. (2015). The last paper focuses on the link between the term structure of interest rates and expected economic growth, also suggesting a role for expected volatilities. However, it predicts a negative correlation between the slope of the yield curve and that of the term structure of volatilities.[2]

Different strands of the literature have investigated the term structure of interest rates, finding strong links between its slope and macroeconomic dynamics. Some papers proposed multi-factor no-arbitrage models enriched by macro variables (Ang and Piazzesi, 2003; Diebold et al., 2006; Hordal et al., 2006; Rudebusch and Wu, 2008), while others studied yield curves in consumption-based models with other preference specifications than "habit" ones (Piazzesi and Schneider, 2007; Rudebusch and Swanson, 2012) or in production-based frameworks (Jermann, 2013; Chen, 2017; Schneider, 2017). However, none of them has made a specific focus on the economic factors behind inversions.[3] By adding a (latent) state factor to Wachter's model as a driver of consumer's volatility perception, we contribute to the habit literature in one of the directions suggested in Cochrane (2016).

This paper is organized as follows: Section 2 describes the benchmark model and presents some empirical findings on the relation between real rates and consumption; Section 3 describes the model of the real short rate with regime switches in the volatility of the surplus-consumption ratio and explains the mechanics of the inversion of the real and nominal term structures; Section 4 describes the empirical analysis and Section 5 concludes.

2 Risk-free rates and consumption growth

In this Section we explain the main arguments that motivate our research. First, we describe the features of the model proposed by Campbell and Cochrane (1999) (CC henceforth), which we take as a benchmark, focusing on the equilibrium risk-free rate; then, we show that the relationship between real short rates and consumption growth is unstable.

[2]This comes from the assumption of CRRA preferences; we adopt habit preferences since this last prediction is at odds with the empirical evidence shown in Figure 1.3.

[3]Chen (2017) proposed a production economy model with external habits that links firm value volatility (due to high adjustment costs) to risk-free rate volatility. In his model, intertemporal substitution and precautionary motives cancel out, leading to a moderately upward-sloping real term structure.

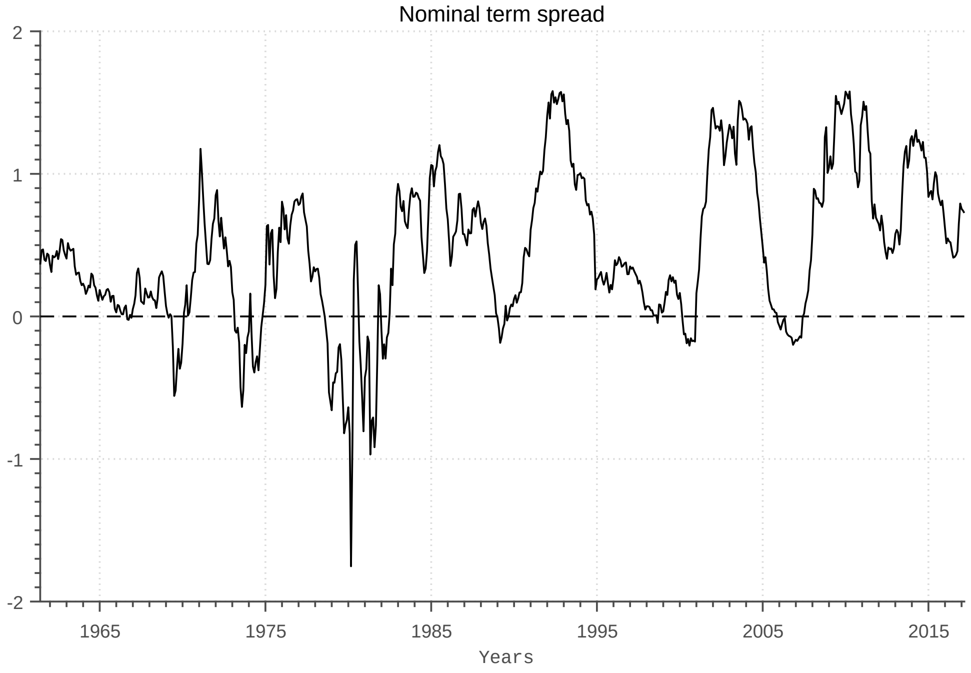

Figure 1.1: Slope of the US Government yield curve: 5-year minus 2-year rates. Monthly averages of daily data. Sample: June 1961 to March 2017. The yields are taken from Gurkaynak et al. (2007).

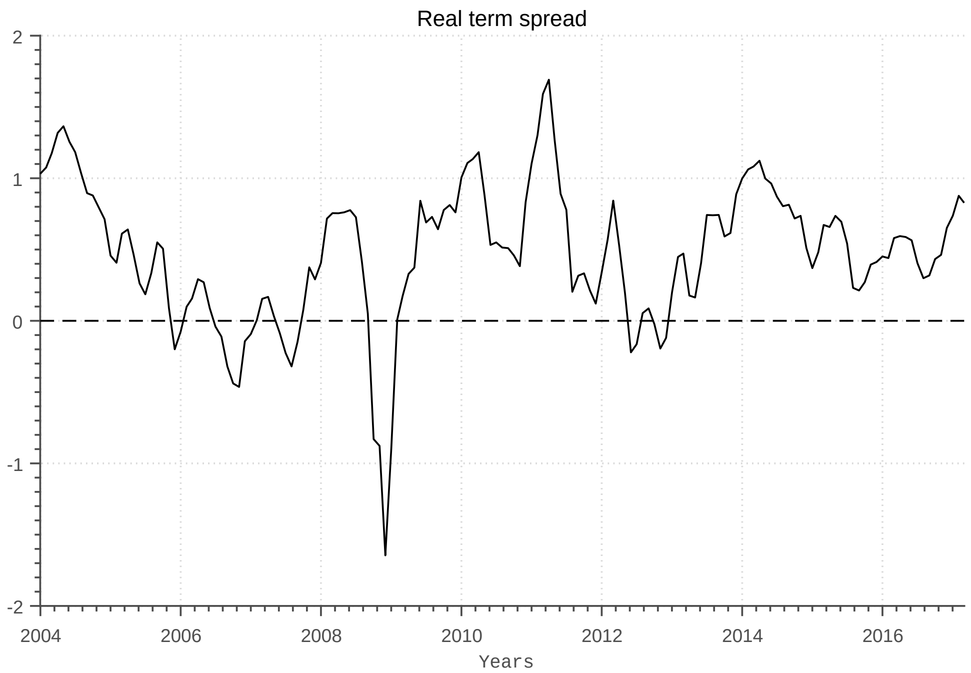

Figure 1.2: Slope of the US TIPS yield curve: 5-year minus 2-year rates. Monthly averages of daily data. Sample: January 2004 to March 2017. The yields are taken from Gurkaynak et al. (2010).

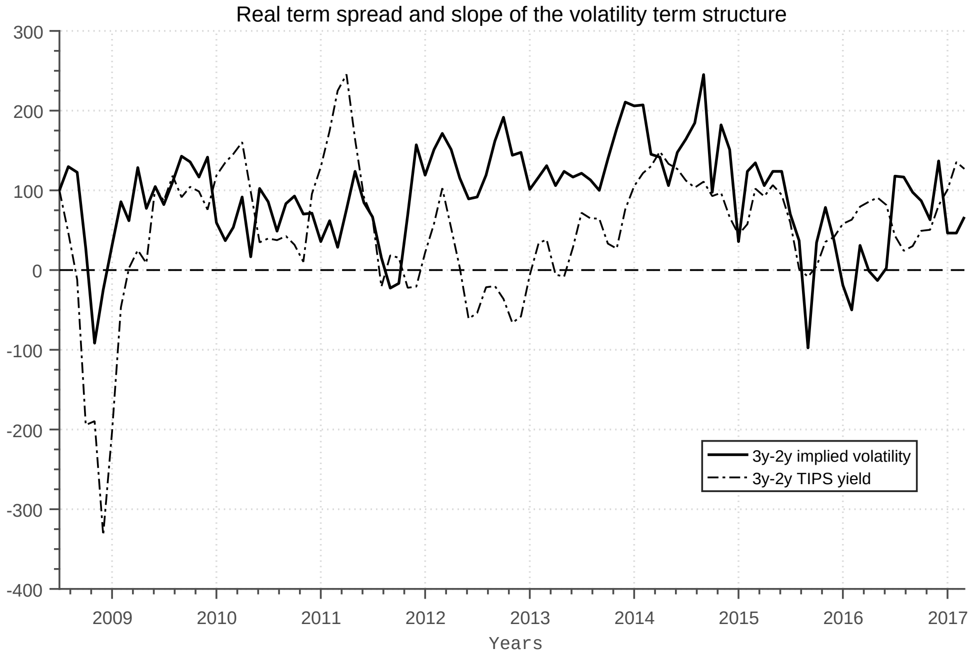

Figure 1.3: Slope of the real yield curve (3-year minus 2-year US TIPS yields) vs. slope of the volatility yield curve (3-year minus 2-year at-the-money implied volatilities from options on S&P500 futures). The choice of the long-term horizon (three years) is due to option data availability. The values of the two slopes are set equal to 100 in June 2008. Monthly averages of daily data. Sample: June 2008 to March 2017. The yields are taken from Gurkaynak et al. (2010).

2.1 Benchmark model

Representative agents have preferences over consumption with respect to a slow-moving reference level X_t, defined as an external habit level:

$$E_t \sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma} \tag{2.1}$$

where β is the subjective time discount factor and γ the utility curvature. To capture the relation between C_t and X_t, CC define the surplus-consumption ratio as the excess consumption relative to habits over the consumption level C_t:

$$S_t = \frac{C_t - X_t}{C_t} \tag{2.2}$$

S_t summarizes all the relevant information on the state of the economy and is the only state variable of the model. Note that $S_t \in [0,1]$; throughout the paper, we will refer to bad states as states characterized by low S_t and to good states as those with S_t close to 1.

Consumers' relative risk aversion is time-varying and countercyclical:

$$\xi_t = \frac{\gamma}{S_t} \tag{2.3}$$

For a constant γ, it falls during booms and increases during recessions.
Consumption growth is exogenous and assumed to follow a random walk

$$\Delta c_{t+1} = g + v_{t+1}, \qquad v_{t+1} \sim N(0, \sigma), \tag{2.4}$$

In order to ensure that consumption always remains above habits, CC define an exogenous process for the log of the surplus-consumption ratio $s_t = ln(S_t)$, which they calibrate in a way that ensures procyclicality: s_t is mean reverting, autoregressive, correlated to shocks to consumption growth and heteroscedastic, with a positive time-varying coefficient $\lambda(s_t)$ loading on the innovation to consumption growth[4]. The term $\lambda(s_t)$ is a sensitivity parameter defined as a square root function of past values of the s_t process. s_{t+1} follows

$$s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)(\Delta c_{t+1} - g) \tag{2.5}$$

where g is the average growth rate of consumption, ϕ the parameter regulating habit persistence, and

$$\lambda(s_t) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2(s_t - \bar{s})} - 1 & \text{if } s \leq s_{max} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

$$s_{max} = \bar{s} + \frac{1}{2}(1 - \bar{S}^2) \tag{2.7}$$

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1 - \phi}} \tag{2.8}$$

As CC show, the functional forms of $\lambda(s_t)$ and $\bar{s} = \ln \bar{S}$ are such that: (i) the risk-free rate is constant; (ii) habit is predetermined at the steady state $s_t = \bar{s}$; (iii) habit is predetermined near the steady state and moves nonnegatively with consumption everywhere[5].

Wachter (2006) applies an alternative specification suggested by CC, which verifies requirements (ii) and (iii) but allows the short-term rate to be a linear function of the state. The functional form of $\lambda(s_t)$ is left unchanged, but $\bar{S}$ is now calibrated in the following

[4] We will refer to s_t or S_t interchangeably as surplus-consumption ratio.

[5] Habit is a non-linear function of consumption. Around the steady state CC prove that $x_t = ln(X_t)$ is such that

$$x_{t+1} \approx (1 - \phi)\bar{x} + \phi x_t + (1 - \phi)c_t$$

way:

$$\bar{S} = \sigma_v \sqrt{\frac{\gamma}{1 - \phi - b/\gamma}} \tag{2.9}$$

From the Euler equation, the real one-period equilibrium risk-free rate is proportional to the deviations of s_t from $\bar{s}$:

$$r_{t,t+1} = \bar{r} - b(s_t - \bar{s}) \tag{2.10}$$

where

$$\bar{r} = -\ln \delta + \gamma g - \frac{\gamma^2 \sigma^2}{2\bar{S}^2} \tag{2.11}$$

and

$$b = \gamma(1 - \phi) - \frac{\gamma^2 \sigma^2}{\bar{S}^2} \tag{2.12}$$

Importantly, since $\{\delta, \gamma, g, \sigma, \phi\}$ are all constant parameters, it follows that b is constant over time. The sign of the b term is key to get the relationship between real rates and surplus-consumption; moreover, it has a clear economic interpretation.

Note that in an asset pricing framework, agents are not allowed to save to shift consumption bundles over time: in equilibrium, asset prices adjust to make them happy to consume the whole endowment in each period. Intertemporal consumption-saving preferences are governed by an intertemporal substitution and a precautionary saving motive. If $b > 0$, then the intertemporal substitution effect dominates: in good times, agents are more willing to save than to consume so the risk-free rate is driven down in equilibrium. On the contrary, if $b < 0$, then the precautionary motive dominates: in good times, agents are less risk-averse, so they would like to borrow to consume more today, driving up the equilibrium interest rate.[6]

In CC's framework, b is set equal to 0 to completely offset these two effects. Instead, Wachter (2006) parameterizes b as a positive constant, so that the inter-temporal substitution effect always wins out, entailing a negative correlation between surplus-consumption and equilibrium interest rates. Note that the b term determines not only the level, but also the slope of the equilibrium term structure of risk-free rates: if $b > 0$, then the dominance of the intertemporal substitution motive is such that, in bad times, agents value consumption today more than consumption tomorrow and the equilibrium term structure is always upward sloping.

We now complete a preliminary analysis by taking a closer look at the relationship between s_t and $r_{t,t+1}$.

[6] In bad times, by contrast, the intertemporal substitution propensity drives the equilibrium interest rate up, while the precautionary saving motive drives it down.

2.2 Real rates and surplus-consumption

We have previously said that in standard consumption-based models featuring habit, the equilibrium real risk-free rate is either constant or a negative function of the surplus-consumption ratio. Assuming government bond yields in the United States to be risk free, we investigate this issue empirically by comparing the historical dynamics of the real rate to that of the surplus-consumption ratio. Real rates - which cannot be proxied by TIPS in this analysis due to data availability - are estimated as the difference between the 3-month T-Bill rate and 3-month expected inflation, with the latter proxied by inflation forecasts made from an estimated autoregressive process (see Appendix A for details); the surplus-consumption ratio is instead constructed as the weighted average of past consumption growth with decreasing weights, as implied by the model and explained in Wachter (2006).[7] Figure 2.1 displays the two series on a quarterly frequency from 1962 to 2014.

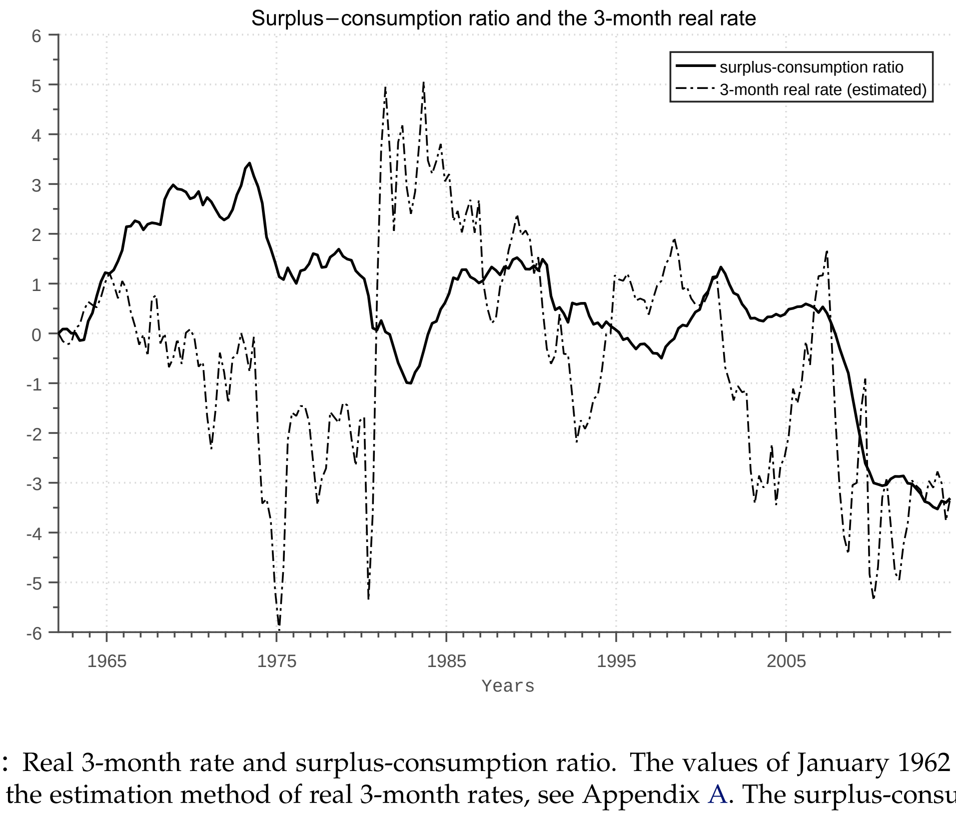

Figure 2.1: Real 3-month rate and surplus-consumption ratio. The values of January 1962 are set equal to zero. For the estimation method of real 3-month rates, see Appendix A. The surplus-consumption ratio is the weighted average of 40 quarters of past consumption growth with decreasing weights, as in Wachter (2006).

[7]While surplus-consumption is theoretically influenced by all its own past values, we choose 40 quarters as the cut-off point. The value of the habit persistence ϕ is as in Table 2. We took this value from Wachter (2006) after we performed a sensitivity analysis for this parameter.

A rapid graphical inspection suggests that the co-movement between the two is not stable over time: the correlation seems positive between the late 1960s and late 1970s, then turns negative during the 80s and 90s, and is unclear for the rest of the sample. To analyze this relationship more formally, we estimate a time-varying b by making rolling regressions on a 10-year window of the real 3-month rate on a constant and on our surplus-consumption proxy. The equation is

$$r_{t,t+1} = a_t + b_t \sum_{j=1}^{40} \phi^j \Delta c_{t-j} + \epsilon_{t+1} \tag{2.13}$$

The rolling estimate is displayed in Figure 2.2. The b coefficient exhibits large time variations, ranging from significantly negative to positive values. This means that real rates depend positively on the surplus-consumption in some part of the sample and negatively in some other parts, a feature of the data which is ruled out in CC's and Wachter (2006)'s models.

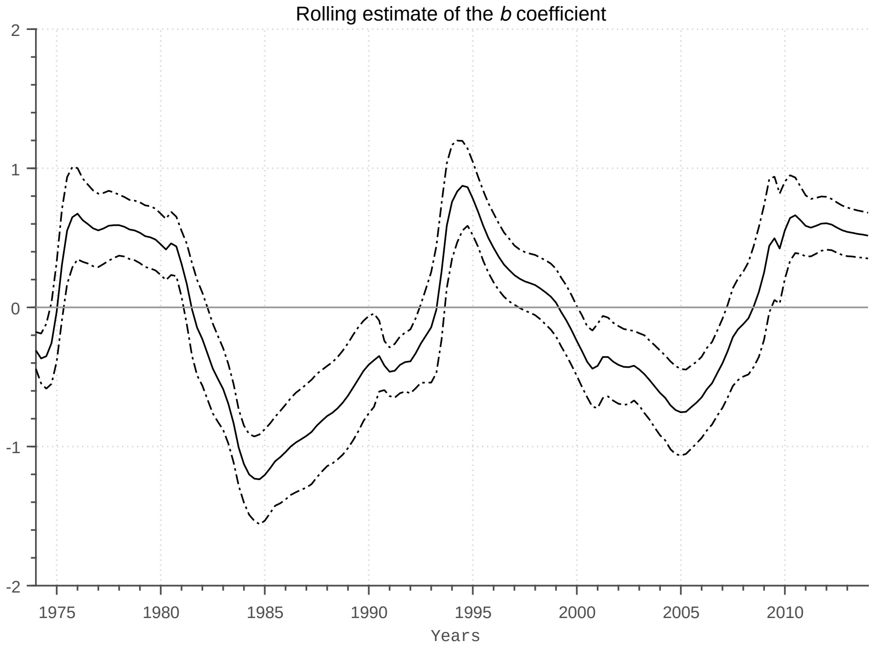

Figure 2.2: Coefficient b of regression $r_t = constant + bs_t + error$, with 95% confidence bands; the real rates r_t are estimated as the difference between the 3-month T-Bill rate and 3-month expected inflation; the surplus-consumption ratio s_t is constructed as the weighted average of past consumption growth with decreasing weights

3 Model

We propose a model that can accommodate the time-varying correlation between the surplus-consumption ratio and risk-free rates, and evaluate its implications for the slope of the term structure. First, we introduce a Markov switching process for consumption growth and derive the pricing equation (Section 3.1); second, we discuss the behavior of the equilibrium risk-free rate and the equilibrium term structure (Section 3.2); third, we include inflation to explain the implications of the model for the nominal yield curve (Section 3.3).

3.1 Markov switching consumption growth and the equilibrium risk-free rate

For the representative agent, we adopt the same set of preferences as CC and maintain the same notation throughout the section:

$$E_t \sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma} \tag{3.1}$$

We assume that, instead of being lognormal, consumption growth is a Markov switching process, in which volatility switches between two regimes.[8] Denoting with g the constant drift, we assume that the process of log consumption growth Δc_{t+1} is

$$\Delta c_{t+1} = g + \sigma_{\zeta_{t+1}} \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0,1) \tag{3.2}$$

with σ_{ζ_t} being either σ_h (high) or σ_l (low), with $\sigma_h > \sigma_l$. Volatility is unobservable, depending on a latent variable ζ_t indicating the state of the economy. Agents infer the state of the economy from observable consumption data. Denote by $\mathbf{P}$ the transition probability of being in state $j = h, l$ coming from state $i = h, l$

$$\mathbf{P} = \begin{bmatrix} p_{hh} & p_{hl} \\ p_{lh} & p_{ll} \end{bmatrix}, \tag{3.3}$$

[8]Given that the trade-off between intertemporal substitution and precautionary saving motives does not depend on the drift of consumption growth, to keep the model as parsimonious as possible we do not impose latent states for it.

which is given and known to the agents at each point in time; new incoming information updates the likelihood of each state

$$\eta_t = \begin{bmatrix} f(\Delta c_t | s_t = 1, \mathbf{X}_{t-1}) \\ f(\Delta c_t | s_t = 2, \mathbf{X}_{t-1}) \end{bmatrix},$$

where $\mathbf{X}_{t-1}$ represents all information at time $t-1$. Then, transition probabilities and updated likelihoods are used to form the posterior probability of being in each state based on the available data. Call $\xi_{t|t-1} \in \mathbb{R}^2$ the posterior belief vector at time $t-1$, Bayes' Law implies that

$$\xi_{t+1|t} = \mathbf{P}' \frac{\xi_{t|t-1} \odot \eta_t}{\mathbf{1}'(\xi_{t|t-1} \odot \eta_t)}$$

where $\odot$ denotes element-by-element product and $\mathbf{1}$ is a 2-by-1 vector of ones.

As consumption growth, the surplus-consumption is also Markov switching:

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t)\sigma_{\zeta_{t+1}}\epsilon_{t+1} \tag{3.4}$$

where ϕ is the AR coefficient (and the habit persistence coefficient). As in CC, habit is a slow moving average of consumption growth and is predetermined at and near the steady state. We keep the same specification for the sensitivity function $\lambda(s_t)$ as in CC and Wachter (2006), i.e. a negative function of s_t: the higher the surplus-consumption, the lower the sensitivity of s to innovations in consumption growth. $\lambda(s_t)$ is inversely proportional to the long-run steady state level $\bar{S}$, that is now computed using the unconditional mean of the stochastic volatility process σ^* and the steady-state level of the b parameter, b^*:

$$\bar{S} = \sigma^* \sqrt{\frac{\gamma}{1-\phi-b^*/\gamma}} \tag{3.5}$$

where

$$\sigma^* = \frac{1}{2}(\sigma_h + \sigma_l)$$

and b^* is such that

$$\bar{r} - b^*(s_{max} - \bar{s}) = 0$$

with $\bar{s} = \ln(\bar{S})$, $\bar{r}$ is the average real risk-free rate and s_{max} as in Equation 2.7. The latter Equation implies that, at the steady state, the risk-free rate is non-negative and the term structure is upward sloping as in Wachter (2006).

The stochastic discount factor (SDF) is a function of the surplus-consumption:

$$M_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t} \frac{S_{t+1}}{S_t} \right)^{-\gamma} = \delta \exp \left\{ -\gamma [g + (1-\phi)(\bar{s} - s_t) + (\lambda(s_t) + 1)\sigma_{\zeta_{t+1}} \epsilon_{t+1}] \right\} \tag{3.6}$$

Solving for the equilibrium risk-free rate involves the computation of the expectation of the SDF as a function of the two stochastic components of s_t, i.e. $\{\epsilon, \zeta\}$. After some algebra, we get

$$r_{t,t+1} = \ln \frac{1}{E_t^{(\epsilon,\zeta)}(M_{t+1})} = -\ln \delta + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \ln E_t^{(\epsilon,\zeta)} \left(e^{-\gamma[\lambda(s_t)+1]\sigma_{\zeta_{t+1}} \epsilon_{t+1}} \right) \tag{3.7}$$

where the last term on the right hand side is

$$-\ln E_t^{(\epsilon,\zeta)} \left(e^{-\gamma[\lambda(s_t)+1]\sigma_{\zeta_{t+1}} \epsilon_{t+1}} \right) = -\ln \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) E_t^{(\epsilon)} \left(e^{-\gamma[\lambda(s_t)+1]\sigma_j \epsilon_{t+1}} | \sigma_{\zeta_{t+1}} = \sigma_j, \xi_{t+1|t} \right) \tag{3.8}$$

Equation 3.8 tells that, in a Markov switching world, agents have expectations about the future draws of consumption - that can be characterized by high or low volatility - and weight them by the posterior probability (i.e., the belief they have at time t) that such draws come from a high or a low volatility state.[9] We interpret this factor as a *precautionary saving effect*, provided that Equation 3.7 differs from the risk-free rate specification in CC's model only for that.[10] In the extreme cases in which $\xi_{t+1|t}(\sigma_h) = 0$ or $\xi_{t+1|t}(\sigma_h) = 1$, the formula for the equilibrium risk-free rate collapses to CC's one.

The previous formula embeds one of the key results of our model: the intensity of the precautionary saving effect depends not only on the current state of the economy, summarized by s_t, but also on an agent's beliefs and, precisely, on the posterior probability attached to the two volatility states. Assume that σ_l is low enough to let the intertemporal substitution effect dominate on the precautionary saving motive, and let σ_h be high enough to allow for the opposite. The steady state volatility level σ^* is such that

[9]Note that the expectation in Equation (3.8) has a closed-form solution. Indeed, Equation (3.8) can be rewritten as

$$-\ln E_t^{(\epsilon,\zeta)} \left(e^{-\gamma[\lambda(s_t)+1]\sigma_{\zeta_{t+1}} \epsilon_{t+1}} \right) = -\ln \sum_{j \in \{h,l\}} \xi_{t+1|t}(j) e^{\frac{\gamma^2}{2}(1+\lambda(s_t))^2 \sigma_j^2}$$

[10]In CC paper the closed-form solution of the risk-free rate is

$$r_{t,t+1} = \ln \frac{1}{E_t M_{t,t+1}} = -\ln \delta + \gamma g + \gamma(\phi - 1)(s_t - \bar{s}) - \frac{\gamma^2 \sigma^2}{2}(1 + \lambda(s_t))^2 \tag{3.9}$$

$\sigma_l < \sigma^* < \sigma_h$, then our model is flexible enough to accommodate both circumstances in which the intertemporal substitution motive prevails over the precautionary motive and times in which the opposite happens. In this model, negative rates and a downward-sloping yield curve appear as a deviation from a positive, upward-sloping steady state.

To summarize, the equilibrium one-period interest rate depends on the combination of the current state and beliefs over the next period consumption. Indeed, states in which s_t is high might no longer be perceived as good states if σ is also expected to be high in the short term: taken s_t as given, when $\xi_{t+1|t}(\sigma_h)$ is higher than $\xi_{t+1|t}(\sigma_l)$, the equilibrium risk-free rate is driven up. Therefore, the combination of high s_t and low $\xi_{t+1|t}(\sigma_h)$ defines good states, while bad states are those with low s_t and high $\xi_{t+1|t}(\sigma_h)$. $\xi_{t+1|t}$ evolves based on the updated likelihood of the two states. Agents have imperfect information on the volatility, so a sequence of large shocks to consumption growth can induce agents to weight more the high volatility state, while a sequence of small shocks slowly pushes them towards the low volatility state.

By introducing Markov switching consumption growth, we allow the trade-off between intertemporal substitution and precautionary saving motives to be endogenous. The flexibility of this specification allows to match the fact that the correlation between real short rates and surplus-consumption is time-varying, and provides a rationale for the periods of positive correlation that appear from the empirical estimation of Equation 2.13.

3.2 The term structure of real risk-free rates

In the previous subsection, we have highlighted the key features underlying this model: time-varying posterior beliefs allow the inter-temporal and precautionary saving motives to dominate at different times, also making time-varying the correlation of r_t with s_t. Let's now turn to the pricing of real risk-free bonds with maturities beyond one period to gain insights into the behavior of the entire term structure of interest rates.

The price at time t of a real bond maturing after n periods ($P_{n,t}$) is computed as the expectation of the future compounded SDFs until maturity. From the Euler equation:

$$\begin{aligned} P_{n,t} &= E_t\left[M_{t+1}P_{n-1,t+1}\right] \\ &= E_t[e^{\ln\delta-\gamma g+\gamma(1-\phi)(s_t-\bar{s})-\gamma[\lambda(s_t)+1]\sigma_{\zeta_{t+1}}\epsilon_{t+1}}P_{n-1,t+1}] \\ &= \sum_{j\in\{h,l\}}\xi_{t+1|t}(j)E_t[e^{\ln\delta-\gamma g+\gamma(1-\phi)(s_t-\bar{s})-\gamma[\lambda(s_t)+1]\sigma_j\epsilon_{t+1}}P_{n-1,t+1}|\sigma_{\zeta_{t+1}}=\sigma_j,\xi_{t+1|t}] \end{aligned} \tag{3.10}$$

with boundary condition $P_{0,t} = 1$; the yield-to-maturity is

$$y_{n,t} = -\frac{1}{n} \ln P_{n,t} \tag{3.11}$$

As described in Equation 3.10, the real bond price is obtained by iterating forward one-period expectations of the bond price for n periods. While future states of the economy are not known at time t, agents can only make expectations conditional on the available information at time t. In order to account for all possible future states for both ϵ and the posterior beliefs ξ for n periods, the bond price is solved numerically on a grid.

As explained in the previous section, if we assume σ_h to be high enough to let the precautionary saving effect dominate, posterior beliefs biased towards σ_h are such that this scenario applies. In these cases, the precautionary saving motive implies agents' willingness to save long-term, because they know that high volatility states have a limited duration and eventually return to the low level: in this case, the 'term structure of volatility beliefs' is downward sloping. Shifting the saving propensity from the short- to the long-run implies that the agent would be more willing to hold long- rather than short-term bonds, therefore in equilibrium short-term securities will pay a premium with respect to long-term ones, entailing the inversion of the equilibrium yield curve.

3.3 Nominal yield curve

Denote by $\pi_t = \ln \Pi_t$ the natural logarithm of the price level and introduce inflation $\Delta\pi_t$ as a first-order autoregressive, exogenous state process (following Cox et al. (1985) and Bekaert et al. (2004)):

$$\Delta\pi_{t+1} = \eta_0 + \psi_0 \Delta\pi_t + \sigma_{\Delta\pi} v_{t+1} \tag{3.12}$$

Denote also by ρ the linear correlation between v_{t+1} and ϵ_{t+1} (i.e., the innovation in consumption growth). We can prove that the nominal bond price is equal to the expected discounted nominal payoff:[11]

$$P^{\$}_{n,t} = E_t \left[M^{\$}_{t+1} P^{\$}_{n-1,t+1} \right] = F^{\$}_n(s_t) e^{A_n + B_n \Delta\pi_t} \tag{3.13}$$

[11] Appendix B reports the proof of the nominal bond pricing formula.

with

$$F_n^{\$}(s_t) = E_t[e^{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}} M_{t+1} F_{n-1}^{\$}(s_{t+1})]$$
$$A_n = A_{n-1} + (B_{n-1}-1)\eta_0 + \frac{1}{2}(B_{n-1}-1)^2\sigma_{\Delta\pi}^2(1-\rho^2)$$
$$B_n = (B_{n-1}-1)\psi_0$$

The SDF of the nominal security ($M^{\$}$) is the ratio between the SDF of the real bond and the one-period gross inflation:

$$M_{t+1}^{\$} = e^{-\Delta\pi_{t+1}} M_{t+1} \tag{3.14}$$

After some algebra, the nominal bond price becomes

$$P_{n,t}^{\$} = const * \sum_{j\in\{h,l\}} \xi_{t+1|t}(j) E_t^{(\epsilon)} \left[M_{t+1} e^{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}} F_{n-1,t+1}^{\$} | \sigma_{\zeta_{t+1}} = \sigma_j, \xi_{t+1|t} \right] \tag{3.15}$$

with

$$const = e^{A_{n-1}+(B_{n-1}-1)(\eta_0+\psi_0\Delta\pi_t)+0.5(B_{n-1}-1)^2\sigma_{\Delta\pi}^2(1-\rho^2)}$$

and

$$M_{t+1} = e^{\ln\delta - \gamma g + \gamma(1-\phi)(s_t-\bar{s}) - \gamma[\lambda(s_t)+1]\sigma_{\zeta_{t+1}}\epsilon_{t+1}}$$

Note that, by assuming correlated innovations of the two state processes, the expected value in Equation 3.15 can be expressed as a function of ϵ only. The yield-to-maturity of the nominal bond is

$$y_{n,t}^{\$} = -\frac{1}{n}\ln P_{n,t}^{\$} \tag{3.16}$$

The nominal bond price has two additional components with respect to the real bond price: a scale factor that depends on inflation volatility (in *const*) and an extra term in the expectation part of Equation 3.15, i.e. $\exp\{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}\}$. The extra term is key to get the intuition for the role of inflation. This term is a positive function of the product between ρ, ψ_0 (through B) and $\sigma_{\Delta\pi}$. If ρ is negative, as reflecting the existing negative correlation between consumption growth and inflation (Wachter, 2006), the extra term adds to the precautionary saving effect in its impact on the level and the slope of the term structure. Indeed, the agents' desire to make precautionary saving/borrowing now depends not only on beliefs of the future consumption volatility states, but also on inflation volatility: the cumulated perceived risk matters for the slope of the nominal equilibrium yield curve in the same way as consumption volatility risk matters for the real one. If $\sigma_{\Delta\pi}$ is sufficiently high, the nominal yield curve can invert even though posterior beliefs are such that the real one is upward sloping.

To complete the description of the model, we also compute the nominal risk premium up to a constant term, which once again depends on surplus-consumption and agents' posterior probabilities:

$$E_t\left(r^{\$}_{n,t+1} - r^{\$}_{1,t+1}\right) = const + E_t\left(\ln F^{\$}_{n-1}(s_{t+1})\right) - \ln F^{\$}_n(s_t) - \\ - \gamma(1-\phi)(\bar{s} - s_t) + \ln \sum_{j\in\{h,l\}} \xi_{t+1|t}(j) e^{\frac{1}{2}(-\gamma[\lambda(s_t)+1]\sigma_j - \rho\sigma_{\Delta\pi})^2} \quad (3.17)$$

The proof of Equation 3.17 is in Appendix C.

4 Empirical analysis

This section covers the application of the model described in Section 3 to US consumption and inflation data. The estimation of the parameters of the Markov switching process is carried out in Section 4.1. We then solve the model and discuss the behavior of the slope of the term structure in Section 4.2. Finally we simulate from the model and report descriptive statistics in Section 4.3.

4.1 Parameter estimation

We estimate the parameters of the Markov switching model by maximum likelihood. Real per capita consumption expenditures on nondurable goods and services are taken from the US Bureau of Economic Analysis. Following Yogo (2006), we restrict our sample to post-1952 data to exclude the exceptionally high consumption growth that followed World War II. The results are reported in Table 1, Panel A; sample data are from 1952Q1 to 2016Q3.

Average consumption growth is estimated at 0.49 per cent per quarter, while volatility equals 0.22 per cent in the low state and 0.56 per cent in the high state (i.e., the latter is 2.5 times higher than the former). The low volatility state is slightly less persistent: the probability that current high consumption growth volatility persists in the next period is 0.93, while for the low volatility state this probability is 0.88. Consumption growth and posterior probabilities are depicted in Figure 4.1.

Data on the monthly CPI index are taken from the Bureau of Labor Statistics database; inflation is constructed as quarter-on-quarter log returns, where quarterly CPI are values of the last month of the quarter. Estimates of the three parameters of the AR(1) process for

Panel A					
Δc	μ	σ_l	σ_h	p_{ll}	p_{hh}
	0.491	0.223	0.556	0.884	0.930
	(0.029)	(0.014)	(0.045)	(0.280)	(0.284)

Panel B			
$\Delta\pi$	η_0	ψ_0	$\sigma_{\Delta\pi}$
	0.265	0.696	0.573
	(0.058)	(0.036)	(0.035)

Table 1: Parameter estimates of the consumption growth and inflation processes. The values are shown in percentage points. Non-annualized quarterly growth rates of consumption are computed using data on real consumption expenditures on nondurable goods and services taken from the US Bureau of Economic Analysis; inflation is constructed as quarter-on-quarter log returns, where quarterly CPIs are values of the last month of the quarter. CPI data are from the Bureau of Labor Statistics.

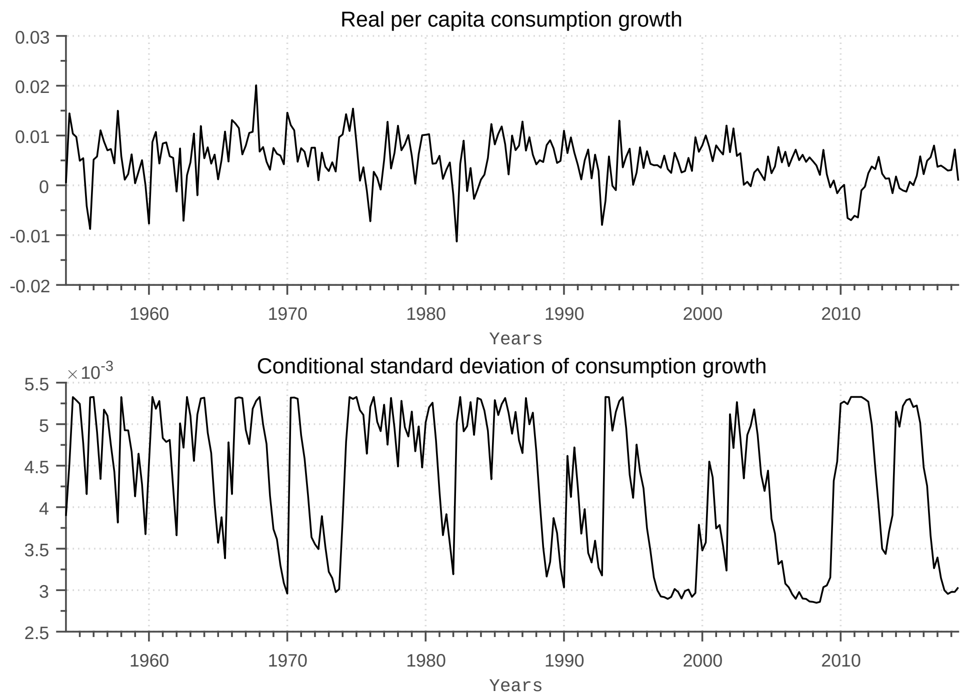

Figure 4.1: Output of the Markov switching estimate. Top panel: real per capita consumption growth. Bottom panel: expected volatility of consumption growth.

inflation are reported in Table 1, Panel B. As functions of those parameters, the long-term mean of the autoregressive process is 0.87 per cent, and its standard deviation is 0.64 per cent, higher than the volatility of consumption growth in the high state. The correlation with consumption growth is estimated at -0.11.

4.2 Model solution

We compute nominal and real bond prices numerically using the series method of Wachter (2005); for this purpose, a quadratic grid constructed as the combination of one grid for s_t and one for $\xi_{t+1|t}$ is employed. Figure 4.2 shows equilibrium real yields in the extreme cases in which the agent perceives with certainty a low or a high volatility in the short term (left and right panel, respectively).

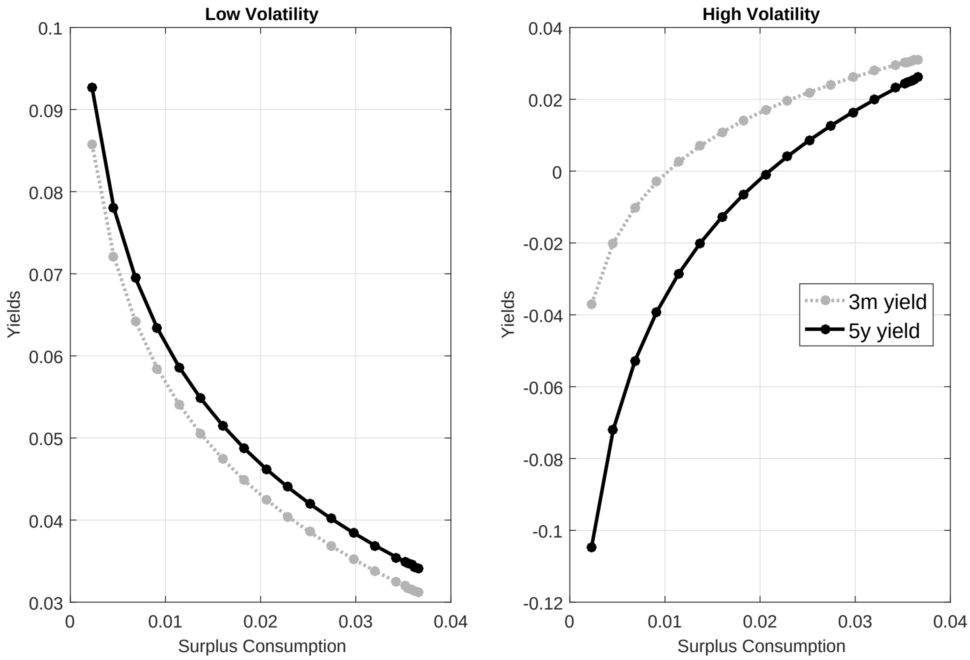

Figure 4.2: Continuously compounded yields on real bonds as a function of the surplus-consumption ratio, implied by the posterior probabilities $P(\sigma = \sigma_h) = 0$ (left panel) and $P(\sigma = \sigma_h) = 1$ (right panel) and the parameters in Table 1 and Table 2. Black line: 5-year real yields; grey line: 3-month real yields.

In the low volatility case, short-term real yields are a decreasing function of the surplus-consumption ratio: real short rates are countercyclical (left panel); moreover, the 5-year yields are always above the 3-month yields, i.e. the equilibrium real term structure is upward sloping for all values of the surplus-consumption ratio. On the contrary, if agents think that in the short-term the volatility of consumption growth will be high (high volatility case, right panel), the precautionary saving motive always prevails: short-term real yields are procyclical and the real term structure is inverted for all values of S_t. Note that the model can account for negative real rates.

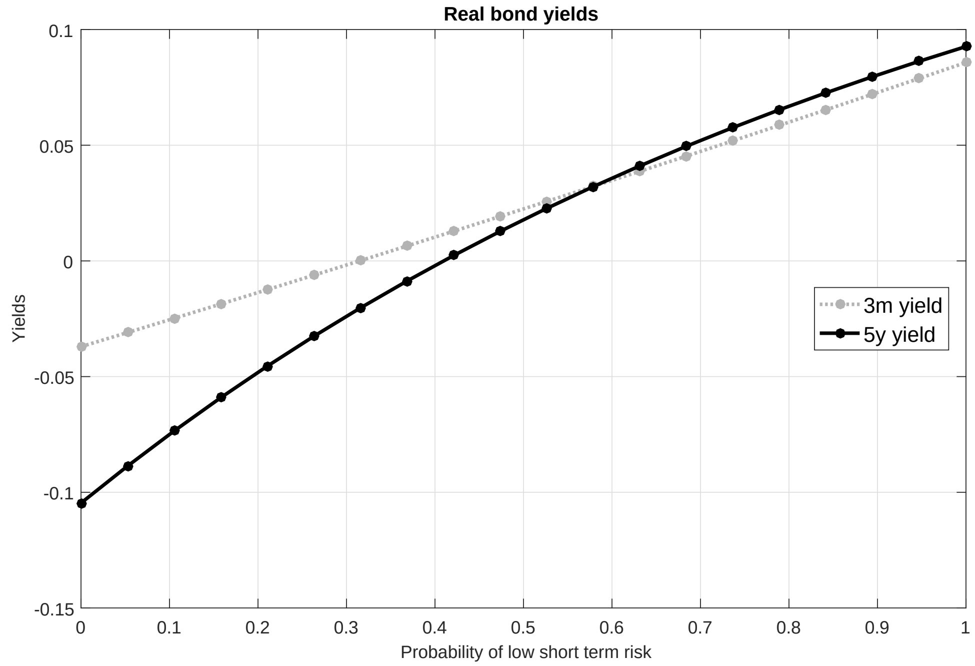

Figure 4.3: Continuously compounded yields on real bonds as a function of the posterior probability to be in the low volatility state in the short term implied by the parameters in Table 1 and Table 2. Black line: 5-year real yields; grey line: 3-month real yields.

Figure 4.3 shows how short- and long-term real yields change as a function of the posterior probability of being in the low volatility state ($P(\sigma = \sigma_l)$) for a given S_t. Both short- and long-term real yields increase with the probability of a low volatility state. The term structure is inverted when the high volatility state is perceived to be more likely (i.e., $P(\sigma = \sigma_l) < 0.5$)), while it is upward sloping in the opposite cases; the threshold value of the probability for which the term structure is inverted depends on the level of surplus-consumption.

We now focus on the nominal curve, studying its sensitivity to the perceived consumption growth volatility across different calibrations of the long-term mean of the inflation process. Figure 4.4 depicts short-term nominal and real yields as a function of the surplus-consumption when the agent expects a low volatility state (left panel) or high volatility state (right panel). We can see that the results on the real risk-free rate carry over to the nominal risk-free rate. Note that nominal yields are always above real yields due to the effect of expected inflation. Figure 4.5 displays 3-month and 5-year nominal yields for different levels of expected inflation when the agent expects a low volatility state (lower panels) or high volatility states (upper panels). We consider expected inflation equal to its

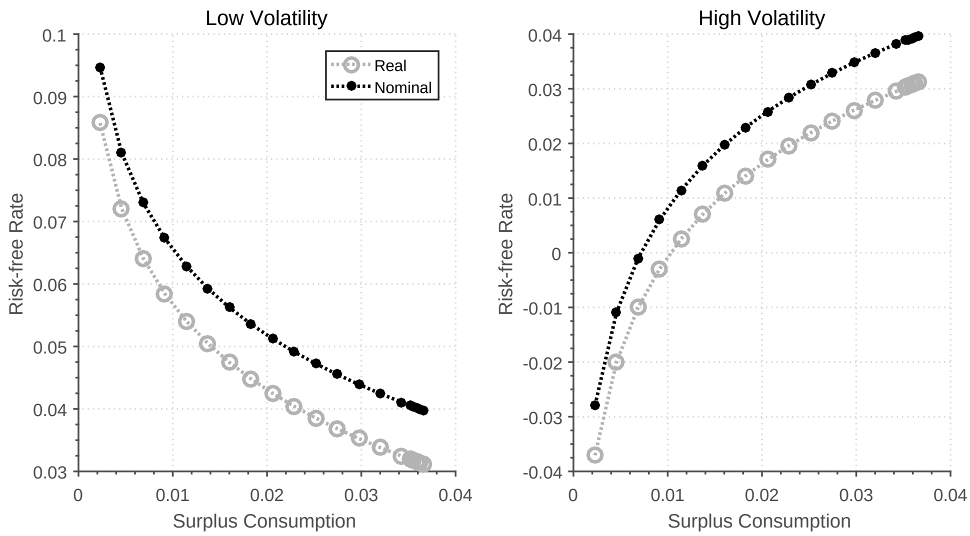

Figure 4.4: Continuously compounded short-term yields on real and nominal bonds as a function of the surplus-consumption ratio implied by the posterior probabilities $P(\sigma = \sigma_h) = 0$ (left panel) and $P(\sigma = \sigma_h) = 1$ (right panel) and the parameters in Table 1 and Table 2. Black line: 3-month nominal yields; grey line: 3-month real yields.

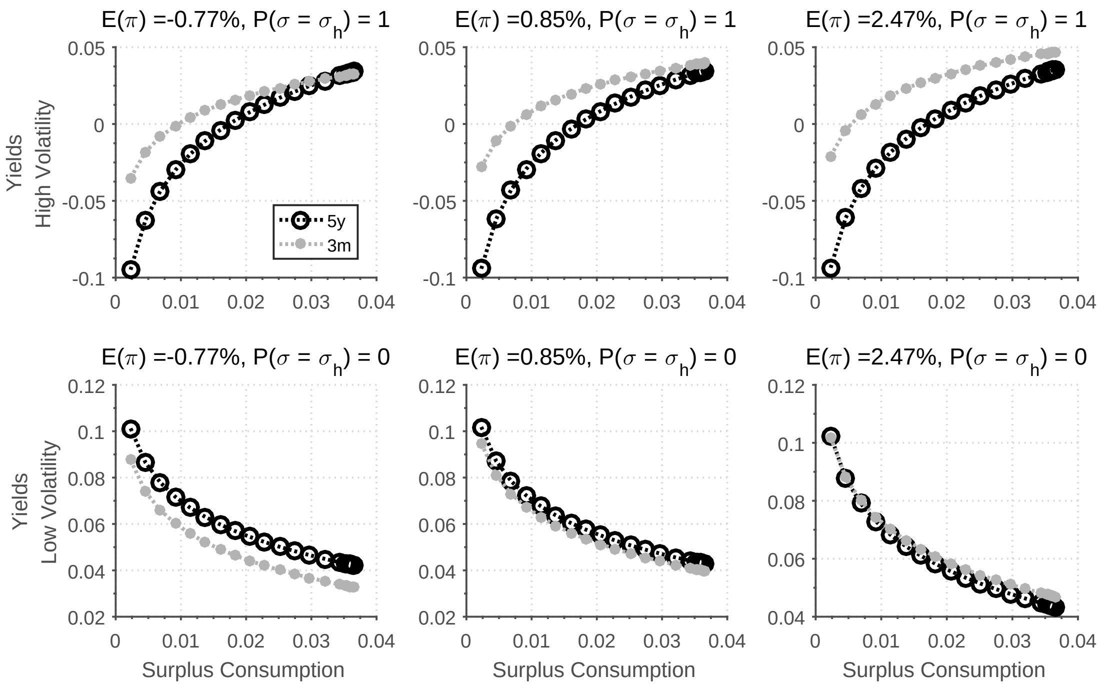

Figure 4.5: Nominal continuously compounded bond yields as a function of the surplus-consumption ratio implied by the posterior probabilities $P(\sigma = \sigma_h) = 1$ (upper panels) and $P(\sigma = \sigma_h) = 0$ (lower panels) and the parameters in Table 1 and Table 2, for different values of expected inflation: long-term expectation (middle panels), long-term expectation minus and plus two standard deviations (left and right panels). Black line: 5-year nominal yields; grey line: 3-month nominal yields.

long-run mean (0.85 per cent, middle panels), and to plus and minus two unconditional standard deviations (right and left panels, respectively).

The equilibrium nominal yield curve is very sensitive to changes in expected inflation. If the agent expects low volatility (lower panels), the higher the long-term inflation expectations, the smaller the difference between long- and short-term yields; in the case of high inflation expectations (lower right panel), this difference is zero or negative for all the levels of surplus-consumption. Provided that inflation expectations are mean reverting, variations in short-term yields are primarily responsible for the inversion. This is consistent with the mechanics explained, in a different setup, by Kurmann and Otrok (2013). If, instead, the agent expects high volatility states (top panels), the nominal yield curve is inverted for all values of the surplus-consumption ratio, except in a few cases in which expectations are deflationary and surplus-consumption is high; moreover, the higher long-term inflation expectations, the larger the gap between long- and short-term yields (top panels, from left to right). This suggests expected inflation is an important driver of the inversion of the nominal term structure.

4.3 Simulation

In order to match the slope of the term structure of nominal interest rates observed in the US market during the sample period, we simulate 100,000 observations of quarterly consumption growth and inflation. The model is calibrated using the parameters in Table 1 and Table 2. Mean and standard deviations of 3-month, 1-year, 3-year and 5-year yields are reported in Figure 4.6.

The model-implied values are very close, on average, to the observed ones. 3-month nominal yields implied by the model are equal, on average, to 5.10 per cent, while the observed ones average at 4.80 per cent; 5-year model-implied and observed nominal yields are equal to 5.89 and 5.91 per cent, respectively. The average positive slope of the time series is therefore matched. Since it is outside of the scope of the paper, we did not include time-varying volatility of inflation in the model; as a consequence, simulated yields are less volatile than market ones.

5 Conclusion

In this paper we propose a consumption-based asset pricing model that allows both the nominal and the real term structure of interest rates to became inverted. The main ingredients are time-varying volatility of consumption growth and imperfect information.

Parameters	Value
Utility Curvature γ	2.00
Habit persistence ϕ	0.97
Derived Parameters	
Discount rate δ	0.95
Long-run mean of log surplus consumption $\bar{s}$	-3.81
Maximum value of log surplus consumption s_{max}	-3.31

Table 2: Assumptions on the parameters of the investor's utility function. The independent parameters (first panel) are set as in Wachter (2006). The second panel gives the derived parameters. The long-run mean of log surplus-consumption, $\bar{s} = ln(S_t)$ in as in Equation 3.5, with $b = 0.0197$ consistent with a risk-free rate of 3.93%, i.e. the average real rate in our sample. The discount rate δ is determined so that, at the steady state, the model-implied nominal risk-free rate equals the one observed in the data. The maximum surplus-consumption ratio is set as in Equation 2.7.

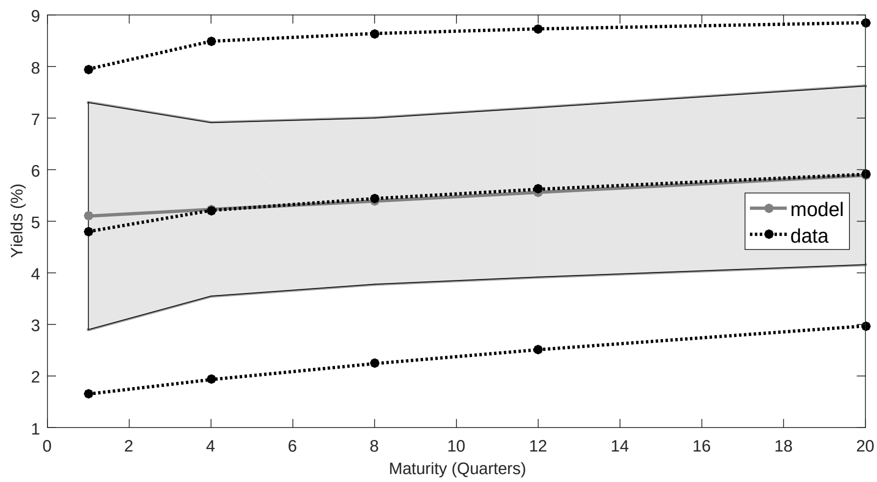

Figure 4.6: Means and standard deviations of continuously compounded zero-coupon bond yields in the model and in the data. 3-month, 1-year, 3-year and 5-year implied yields are compared with data from 1952Q1 to 2016Q3.

Agents form posterior beliefs over future states of the economy. A high expected risk in the short term makes a risk averse agent shift her saving propensity from the short- to the long-run, implying that she would be more willing to hold long- rather than short-term bonds and entailing the inversion of the equilibrium yield curve.

The proposed stochastic discount factor could, in principle, be used to price other types of assets. The impact of macroeconomic risk on equity pricing is investigated by Lettau et al. (2008), among others. The application for corporate bond pricing or derivative pricing can be one avenue of future research. This model is designed for default-free economies: another interesting avenue of research could be that of investigating the evolution of a bond term structure containing a risk premium related to the default of the bond's issuer. Equilibrium yield curves of different countries with different default risks could in this way be compared.

A Market-implied real interest rates

Professional forecasters only started to produce estimates of CPI inflation expectations in the early 1980s, so these cannot be used to retrieve real rates (by subtracting inflation expectations from nominal rates) before that date. We follow instead the procedure proposed in Chapter 3 of April 2014's *World Economic Outlook* published by the IMF: inflation expectations are computed as out-of-sample forecasts from a simulated autoregressive process of inflation. In this way we can estimate real rates for the whole sample (up to the 1960s).

Denoting P_t the monthly consumer price index at time t, an autoregressive model with 12 lags ($AR(12)$) is fitted on the variable $\gamma_t = lnP_t - lnP_{t-12}$; the estimation is carried out on a rolling window of 60 months to mitigate the effect of parameter instability. Model-based inflation expectations for horizon j are computed using out-of-sample forecasts of γ_t. Real rates are then recovered as

$$r_{n,t} = r^{\$}_{n,t} - \frac{(1-g)}{(1-g^n)} \sum_{i=1}^{n} g^i E_t \pi_{t,t+1}$$

where $r_{n,t}$ and $r^{\$}_{n,t}$ are the real and nominal rates at time t on a bond with maturity n, $E_t\pi_{t,t+i}$ is the inflation expectation at time t for period $t+i$ and $g = (1+\bar{r^{\$}})^{-i}$, with $\bar{r^{\$}}$ being the average nominal rate. The real rate is therefore equal to the nominal rate minus a weighted average of the inflation expectation over the entire life of the bond.

B Pricing of real and nominal bonds

Let $P_{n,t}$ denote the price of a real bond maturing in n periods, and $P^{\$}_{n,t}$ the price of a nominal bond. Prices are computed as expectations of the future compounded SDFs until maturity.

The real price is determined recursively from the Euler equation (3.10) with boundary condition $P_{0,t} = 1$. Note that $P_{n,t}$ is a function of the posterior probability $\xi_{t+1|t}$. We solve for these functional equations numerically on a grid of values for the state variable $\xi_{t+1|t}$. Conditional on $\xi_{t+1|t}$, the price of the bond is a function of s_t alone, so equation (3.10) can be rewritten as

$$\begin{aligned}
P_{n,t} &= E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\frac{S_{t+1}}{S_t}\right)^{-\gamma}P_{n-1,t+1}\right] \\
&= E_t\left[M_{t+1}P_{n-1,t+1}\right] \\
&= \sum_{j\in\{h,l\}}\xi_{t+1|t}(j)E_t[M_{t+1}P_{n-1,t+1}|\sigma_{\zeta_{t+1}}=\sigma_j,\xi_{t+1|t}] \\
&= \sum_{j\in\{h,l\}}\xi_{t+1|t}(j)E_t[e^{\ln\delta-\gamma[g+(1-\phi)(\bar{s}-s_t)+(\lambda(s_t)+1)\sigma_j\epsilon_{t+1}]}P_{n-1,t+1}|\sigma_{\zeta_{t+1}}=\sigma_j,\xi_{t+1|t}]
\end{aligned}$$

The last expectation can be solved using numerical integration on a grid of values for s_t, conditional on being in state j.

Analogously, the nominal bond price is equal to the expected discounted nominal payoff:

$$P^{\$}_{n,t} = E_t[M_{t+1}\frac{\Pi_t}{\Pi_{t+1}}P^{\$}_{n-1,t+1}] \tag{B.1}$$

In order to compute the nominal bond prices we introduce inflation as an additional state variable. Using the law of iterated expectations and conditioning on realizations of the shock to the level of the consumption growth, we can prove that

$$P^{\$}_{n,t} = F^{\$}_{n,t}\exp\{A_n + B_n\Delta\pi_t\} \tag{B.2}$$

with

$$\begin{aligned}
F^{\$}_{n,t} &= E_t[M_{t+1}\exp\{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}\}F^{\$}_{n-1,t+1}] \\
A_n &= A_{n-1} + (B_{n-1}-1)\eta_0 + 0.5(B_{n-1}-1)^2\sigma^2_{\Delta\pi}(1-\rho^2) \\
B_n &= (B_{n-1}-1)\psi_0
\end{aligned}$$

The boundary conditions are $F^{\$}_{0,t}=1$, $A_0=0$, and $B_0=0$.

The proof is by induction. Suppose equation (B.2) is true for $P^{\$}_{n-1,t+1}$. Then, from the Euler equation it must be that

$$\begin{aligned}
P^{\$}_{n,t} &= E_t[M_{t+1}\frac{\Pi_t}{\Pi_{t+1}}\exp\{A_{n-1}+B_{n-1}\Delta\pi_{n+1}\}F^{\$}_{n-1,t+1}] \\
&= E_t[M_{t+1}\exp\{-\eta_0-\psi_0\Delta\pi_t-\sigma_{\Delta\pi}v_{t+1}+A_{n-1}+B_{n-1}(\eta_0+\psi_0\Delta\pi_t+\sigma_{\Delta\pi}v_{t+1})\}F^{\$}_{n-1,t+1}] \\
&= \exp\{A_{n-1}+(B_{n-1}-1)(\eta_0+\psi_0\Delta\pi_t)\}E_t[M_{t+1}F^{\$}_{n-1,t+1}\exp\{(B_{n-1}-1)\sigma_{\Delta\pi}v_{t+1}\}]
\end{aligned}$$

If we use the law of iterated expectations twice and condition on $\xi_{t+1|t}$, that is the posterior probability at time $t+1$, and then on ϵ_{t+1}, that is the error on the level of consumption growth we have

$$
\begin{aligned}
P^{\$}_{n,t} =& \exp\{A_{n-1} + (B_{n-1}-1)(\eta_0 + \psi_0 \Delta\pi_t) \\
& \sum_{j\in\{h,l\}} \xi_{t+1|t}(j) E_t[M_{t+1} F^{\$}_{n-1,t+1} \exp\{(B_{n-1}-1)\sigma_{\Delta\pi} v_{t+1}\} | \sigma_{\zeta_{t+1}} \epsilon_{t+1}, \sigma_{\zeta_{t+1}} = \sigma_j, \xi_{t+1|t}]
\end{aligned}
$$

given that

$$
(B_{n-1}-1)\sigma_{\Delta\pi} v_{t+1} | \sigma_j \epsilon_{t+1} \sim N(\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}, (B_{n-1}-1)^2 \sigma^2_{\Delta\pi}(1-\rho^2))
$$

we have

$$
\begin{aligned}
P^{\$}_{n,t} =& \exp\{A_{n-1} + (B_{n-1}-1)(\eta_0 + \psi_0 \Delta\pi_t) + 0.5(B_{n-1}-1)^2 \sigma^2_{\Delta\pi}(1-\rho^2)) \\
& \sum_{j\in\{h,l\}} \xi_{t+1|t}(j) E_t[M_{t+1} F^{\$}_{n-1,t+1} \exp\{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}\} | \sigma_{\zeta_{t+1}} = \sigma_j, \xi_{t+1|t}]
\end{aligned}
$$

Therefore, equation (B.2) is satisfied with

$$
\begin{aligned}
F^{\$}_n(s_t) &= E_t[M_{t+1} \exp\{\rho(B_{n-1}-1)\sigma_{\Delta\pi}\epsilon_{t+1}\} F^{\$}_{n-1,t+1}] \\
A_n &= A_{n-1} + (B_{n-1}-1)\eta_0 + 0.5(B_{n-1}-1)^2 \sigma^2_{\Delta\pi}(1-\rho^2) \\
B_n &= (B_{n-1}-1)\psi_0
\end{aligned}
$$

C Nominal risk premium

Let's compute the nominal risk premium

$$
E_t\left(r^{\$}_{n,t+1} - r^{\$}_{1,t+1}\right) \tag{C.1}
$$

Using formula (3.13) we have that

$$
\begin{aligned}
E_t\left(r^{\$}_{n,t+1}\right) &= E_t\left(\ln F^{\$}_{n-1}(s_{t+1}) + A_{n-1} + B_{n-1}\Delta\pi_{t+1} - \ln F^{\$}_n(s_t) + A_n + B_n \Delta\pi_t\right) = \\
&= cost + E_t\left(\ln F^{\$}_{n-1}(s_{t+1})\right) - \ln F^{\$}_n(s_t) + B_{n-1}\underbrace{(\eta_0 + \psi_0\Delta\pi_t)}_{E_t(\Delta\pi_{t+1})} - B_n\Delta\pi_t = \\
&= cost + E_t\left(\ln F^{\$}_{n-1}(s_{t+1})\right) - \ln F^{\$}_n(s_t) + \psi_0 \Delta\pi_t
\end{aligned}
$$

where the last equality comes from $B_n = (B_{n-1} - 1)\psi_0$.
For the second term, we know that $r^{\$}_{1,t+1} = 1/\ln(M^{\$}_{t+1})$ and

$$E_t\left(M^{\$}_{t+1}\right) = E_t\left(e^{-\Delta\pi_{t+1}} M_{t+1}\right) = \\ = E_t[e^{-(\eta_0+\psi_0\Delta\pi_t+\sigma_{\Delta\pi}v_{t+1})} e^{\ln\delta-\gamma[g+(1-\phi)(\bar{s}-s_t)+(\lambda(s_t)+1)\sigma_{\zeta_{t+1}}\epsilon_{t+1}]}]$$

Using the same methodology that we applied for the formula of the nominal bonds, we have

$$E_t\left(M^{\$}_{t+1}\right) = \exp(\ln\delta - \gamma(g + (1-\phi)(\bar{s} - s_t)) - \eta_0 - \psi_0\Delta\pi_t + 0.5\sigma^2_{\Delta\pi}(1-\rho^2)) \\ \sum_{j\in\{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t)+1)\sigma_j - \rho\sigma_{\Delta\pi})^2)$$

so

$$r^{\$}_{1,t+1} = 1/\ln(M^{\$}_{t+1}) = \\ = -\ln\delta + \gamma(g + (1-\phi)(\bar{s} - s_t)) + \eta_0 + \psi_0\Delta\pi_t - 0.5\sigma^2_{\Delta\pi}(1-\rho^2)) - \\ - \ln\Big(\sum_{j\in\{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t)+1)\sigma_j - \rho\sigma_{\Delta\pi})^2)\Big)$$

Therefore the nominal risk premium is

$$E_t\left(r^{\$}_{n,t+1} - r^{\$}_{1,t+1}\right) = cost + E_t\left(\ln F^{\$}_{n-1}(s_{t+1})\right) - \ln F^{\$}_n(s_t) - \\ - \gamma(1-\phi)(\bar{s} - s_t) + \ln\Big(\sum_{j\in\{h,l\}} \xi_{t+1|t}(j) \exp(0.5(-\gamma(\lambda(s_t)+1)\sigma_j - \rho\sigma_{\Delta\pi})^2)\Big) \tag{C.2}$$

References

Ang, A. and M. Piazzesi (2003), "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables." *Journal of Monetary Economics*, 50(4), 745–787.

Bansal, R. and M. Yaron (2004), "Risks for the long run: a potential resolution of asset pricing puzzles." *Journal of Finance*, 59, 1481–1509.

Bekaert, G., E. Engstrom, and S.R. Grenadier (2004), "Stock and bond returns with moody investors." *Journal of Empirical Finance*, 17, 867–894.

Boguth, O. and L-A Kuehn (2013), "Consumption volatility risk." *Journal of Finance*, LXVIII (6), 2589–2615.

Breeden, D. T., R. H. Litzenberger, and T. Jia (2015), "Consumption-based asset pricing, part 1: Classic theory and tests, measurement issues, and limited participation." *Annual Review Financ. Econ.*, 7, 35–83.

Campbell, J.Y. and H.J. Cochrane (1999), "By force of habit: A consumption-based explanation of aggregate stock market behavior." *The Journal of Political Economy*, 107, 205–251.

Chen, A. Y. (2017), "External habit in a production economy: A model of asset prices and consumption volatility risk." *Review of Financial Studies*, 30(8), 867–894.

Cochrane, J. (2016), "The habit habit." *Hoover institution Economics Working Papers*, 16105.

Cox, J., M. Ingersoll, and S. Ross (1985), "A theory of the term structure of interest rates." *Econometrica*, 53, 385–408.

Diebold, F. X., G. D. Rudebusch, and S. B. Aruoba (2006), "The macroeconomy and the yield curve: a dynamic latent factor approach." *Journal of Econometrics*, 131(1–2), 309–338.

Gurkaynak, R. S., B. Sack, and J. H. Wright (2007), "The u.s. treasury yield curve: 1961 to the present." *Journal of Monetary Economics*, 54(8), 2291–2304.

Gurkaynak, R. S., B. Sack, and J.H. Wright (2010), "The tips yield curve and inflation compensation." *American Economic Journal: Macroeconomics*, 2(1), 70–92.

Hordal, P., O. Tristani, and D. Vestin (2006), "A joint econometric model of macroeconomic and term-structure dynamics." *Journal of Econometrics*, 131(1–2), 405–444.

Jermann, U. J. (2013), "A production-based model for the term structure." *Journal of Financial Economics*, 109(2), 293–306.

Kurmann, A. and C. Otrok (2013), "News shocks and the slope of the term structure of interest rates." *American Economic Review*, 103(6), 2612–2632.

Lettau, M., S.C. Ludvigson, and J. Wachter (2008), "The declining equity premium: What role does macroeconomic risk play?" *Review of Financial Studies*, 21(4).

Piazzesi, M. and M. Schneider (2007), "Equilibrium yield curves." *NBER Macroeconomics Annual 2006*, 21, 389–472.

Rudebusch, G. D. and E. T. Swanson (2012), "The bond premium in a dsge model with long-run real and nominal risks." *American Economic Journal: Macroeconomics*, 4(1), 105–143.

Rudebusch, G. D. and T. Wu (2008), "A macro-finance model of the term structure, monetary policy and the economy." *The Economic Journal*, 118, 906–926.

Schneider, A. (2017), "Risk sharing and the term structure of interest rates." *Job Market Paper, UCLA*.

Wachter, J.A. (2005), "Solving models with external habit." *Finance Research Letters*, 2, 210–226.

Wachter, J.A. (2006), "A consumption-based model of the term structure of interest rates." *Journal of Financial Economics*, 79, 365–399.

Yogo, M. (2006), "A consumption-based explanation of expected stock returns." *Journal of Finance*, 61(2), 539–580.

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Adriana Grasso
LUISS Guido Carli, Rome, Italy; email: grassoa@luiss.it

Filippo Natoli
Bank of Italy, Rome, Italy; email: filippo.natoli@bancaditalia.it

Postal address 60640 Frankfurt am Main, Germany
Telephone +49 69 1344 0
Website www.ecb.europa.eu

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