Short-term forecasting of business cycle turning points:
a mixed-frequency Markov-switching dynamic factor analysis

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Abstract

We develop a Bayesian analysis for the nowcasting and forecasting of turning points in the business cycle using large-dimensional mixed-frequency dynamic factor models for which the means of the latent factors are subject to Markov-switching. The key novelty in the analysis is our model-based treatment of the dynamics in a mixed-frequency data set. In particular, we propose a coincident index based on a quarterly measure of real gross domestic product and on the four monthly economic variables that are commonly used to track the business cycle. We present in-sample and real-time recessions probabilities for the U.S. economy. We show that the inclusion of a quarterly measure significantly improves the real-time estimates of turning points when the NBER recession dates are assumed correct.

1 Introduction

It is of key importance for any economic policy maker to have an accurate and promptly indication of the current economic situation and to have a short-tem forecast for the direction of economic activity. The information is even more valuable when a probability can be provided for an upcoming economic recession. It is not surprising that, after the recent financial and credit crises, new research interest has emerged on the improvement of such indicators and forecasts in the econometric literature. It coincides with the surge of data technology so that many more data sets have become available for empirical research.

However, as pointed out by Hamilton (2011), when dealing with real-time data, the continuous changes in the economic relationships and the existence of subsequent data revisions, transforms the question of tracking the business cycle in a very challenge and difficult task. Of course, when trying to predict the evolution of the economy, characterizing the business cycles is crucial. The seminal work of Burns and Mitchell (1946) is, therefore, a good starting point where many economist have stood. Two key features of the business cycles are highlighted there: co-movements among economic series and the non-linear behaviour of the economy between recession and expansion periods. Stock and Watson (1989, 1991, 1993) handle the first characteristic by proposing a dynamic factor model able to capture unobserved co-movements between economic time series. The second feature was approached by Hamilton (1989) through a univariate two-state regime Markov-switching model where the evolution of the GDP is split between recession and expansion phases. Handling both characteristics within a single unified model was first proposed by Diebold and Rudebusch (1996) and accomplished by both Chauvet (1998) and Kim and Nelson (1998) adopting a multivariate dynamic factor Markov-switching (DFMS) model. However, those models do not included real GDP measures among the economic variable indicators used. The importance of GDP when analyzing the business cycles was highlighted by Stock and Watson (1999) whom stated that "fluctuations in aggregate output are at the core of the business cycle, so the cyclical component of real GDP is a useful proxy for the overall business cycle". Following this concern, Camacho et al. (2012) extends the DFMS model in order to deal with mixing frequencies and incorporate the very important publicly available information of real GDP.

In the present paper, we base on Camacho et al. (2012) and we propose an alternative way of estimation for the mixing-frequencies DFMS model which easily deals with some of the major difficulties the mentioned authors met. Within a linear framework, Mariano and Murasawa (2003) developed a methodology which incorporates mixing-frequencies to dynamic factor models. Their idea is that quarterly GDP series can be interpolated to estimate the unobserved latent monthly GDP through a filtering algorithm. As mentioned by Camacho et al. (2012), extending their framework to the non-linear case brings the "curse of dimensionality problem" since, in order to compute the likelihood for the mixing-frequency DFMS model, 32 parallel Kalman filters have to be evaluated in the most simple case. To solve this problem, the authors propose approximating the 32 states to a simpler version where only 2 different states are taking into account. Through a Monte Carlo experiment, Camacho et al. (2012) show the approximation have small effects in terms of the quadratic probability score when the idiosyncratic variance of the quarterly indicator is high enough.

An alternative way of dealing with mixing-frequencies in the context of a linear multivariate dynamic factor model has recently been proposed by Blasques et al. (2014). In their work, it is shown that using a full system approach where "the higher frequency data is stacked into a vector of observations which has the lower frequency", it is possible to avoid any kind of interpolation while all information incorporated at different frequencies is fully conserved. Following their proposal, we have adapted the "stacked" model to a non-linear framework. Under this setting we overcome the necessity of dealing with the high number of possible paths (or states) Camacho et al. (2012) has to confront with.

A second contribution of our work is proposing a simpler way of estimating a mixing-frequency dynamic factor model through Bayesian methods, both within a linear and a non-linear framework. Marcellino et al. (2013) were the first in develop a Bayesian methodology approach to a linear mixing-frequency dynamic factor model. In their setting, a model similar to the one proposed by Mariano and Murasawa (2003) which incorporate stochastic volatility is estimated. However, their estimation procedure rely on the possibility of estimating the variance covariance matrix of a MA(4) regression equation where the errors follows –in turn– an AR(p) process. This difficulty must be sorted in order to compute the factor loadings associated with low frequency variables (see Section 3.1). When the "stacking" approach is used instead, the problem is avoided since any missing observation exists within quarters for the case of quarterly variables. Loosely speaking, our Bayesian approach can be viewed as an extension of Kim and Nelson (1998) for the mixing frequency case.

Three different –but related– empirical applications for the US economy are carried on using the model here proposed. Following the existing literature, we stands on the four monthly coincident indicator commonly used when analyzing the US business cycles and we add the real GDP measure to estimate our "stacked" mixing-frequency DFMS model. The first exercise done is an in-sample estimation of the recession probabilities for the US economy from 1959 to 2014. Comparing results of using a non-mixing and a mixing-frequency DFMS model we found that adding the GDP data does not significantly increase the identification of recession phases. This is in line with Camacho et al. (2014), which indicates that when the number of high quality data already included is high enough, an additional high quality indicator does not generate large improvements regarding business cycle identification.

Our second exercise consist on a real-time experiment aiming to obtain which would have been the business cycles turning points pointed out by the "stacked" mixing-frequency DFMS model. Once again, results are compared with the non-mixing frequency case. Moreover, we follow Camacho et al. (2012) proposal for dealing with unbalanced panel data in the context of DFMS models and we compare the outcomes of using latest available (non-revised) data and when using a balanced panel data (revised and better quality data). With respect to the inclusion of GDP measure among the indicators, we found the addition means a better accuracy in the real-time distinction of turning points when the NBER recession dates are taken as being the true ones. Using revised or non-revised data does not affects the turning point dates we found. However, it does affects the timing when the model would have announced that a new turning point had appeared. In general, the announcements would have been done one month in advance with respect to balanced panel data case. This result confirms Camacho et al. (2012) findings regarding the necessity of using latest available information when the goal is a sooner identification of turning points.

Our last empirical exercise is a real-time nowcasting experiment where we compare the accuracy of our model with respect to the simple AR(2) model. Our results show using the mixing-frequency DFMS model significantly improves the nowcast accuracy when at least one month of monthly data from the current quarter is already released (i.e. during the second and third month of the quarter). Gains from using DFMS models in forecasting analysis were also found in Chauvet and Potter (2012) in the non-mixing frequency context and by Camacho et al. (2013) in the mixing-frequency one.

From a methodological and an empirical point of view, the "stacked" mixing-frequency DFMS model appears to be a good alternative when a promptly and accurate evaluation of the current and the short-run future state of the economy wants to be achieved.

The paper is structured as follows. In the next section we present the "stacking" approach for the state space representation and we show how to adapt the model for the non-linear case. In Section 3 we describe the Bayesian estimation procedure. The empirical part of the paper is presented in Section 4 which is divided in three parts: subsection 4.1 shows in-sample estimations, subsection 4.2 describe real-time estimation of turning-points and subsection 4.3 is about the nowcasting exercise. Section 5 concludes.

2 The Model

2.1 Single-Index DFMS Model

Following Chauvet (1998) and Kim and Nelson (1998), we represent co-movements among economic variables and business cycle asymmetries within a single statistical model. In this sense, the source of variation of any observable variable (y_{it}) is assumed to be explained by both, an unobserved common factor (f_t) and by an idiosyncratic component (u_{it}). Business cycles shifts are then incorporated by assuming the dynamics of f_t is affected by the unobserved regime-switching state of the economy. Under this setting, observable variables can be represented as,

$$\Delta y_{it} \;=\; \beta_i(L)\Delta f_t + u_{it} \qquad i = 1, \ldots, n \tag{1}$$

where Δy_{it} is the log of the first difference of each observable $i = 1, \ldots, n$[1] while $\beta_i(L)$ is a polynomial lag operator meaning each observable may depend on current and lagged values of the growth rate of the unobserved common component (Δf_t).

The dynamics of the model is given by,

$$\Phi_f(L)\left(\Delta f_t - \mu_{s_t}\right) \;=\; \eta_t \qquad \eta_t \overset{iid}{\sim} N\left(0, \sigma^2_\eta\right) \tag{2}$$

$$\Phi_i(L)u^i_t \;=\; \epsilon_{it} \qquad \epsilon_{it} \overset{iid}{\sim} N\left(0, \sigma^2_i\right) \tag{3}$$

where η_t and ϵ_{it} are independent of each other for all t and i. $\Phi_f(L)$ and $\Phi_i(L)$ are finite lag polynomials governing the autoregressive process' order for the common component (p_f) and for the idiosyncratic (p_i). Business cycles shifts are introduced as a switching mean in the factor ($\Phi_f(L)\mu_{s_t}$), which affects the dynamic of the common factor depending not only on whether the economy is currently in a recession ($S_t = 0$) or in an expansion phase ($S_t = 1$) but also on the lagged states $S_{t-1}, \ldots, S_{t-p_f}$[2]. Transitions between those two states of the economy are in turn described by a first-order Markov-Switching process. Therefore,

$$\mu_{s_t} \;=\; \mu_0 + \mu_1 S_t \qquad \mu_1 > 0\,,\ S_t = \{0, 1\} \tag{4}$$

$$p_{ij} \;=\; Pr[S_t = j | S_{t-1} = i] \qquad \sum_{j=1}^{2} p_{ij} = 1 \quad \forall i \tag{5}$$

where p_{ij} is the transition probability the economy moves from state i in $t-1$ to state j in t. This model can be easily cast under the state-space representation and estimated using the approximate maximum-likelihood estimation method (Chauvet (1998)) or through Bayesian inference (Kim and Nelson (1998)) among other methods.

2.2 Incorporating mixing frequencies

2.2.1 Camacho et al. (2012)

In a recent work, Camacho et al. (2012) proposed extending a similar model to the one described above in order to deal with mixing frequencies (monthly and quarterly variables).

[1] Observable variables are expressed as deviations from means.

[2] Chauvet (1998) assumes instead a switching intercept. In particular, equation (2) is written as $\Phi_f(L)\Delta f_t - \mu_{s_t} = \eta_t$, meaning only current state of the economy affects the common factor dynamics.

In particular, they based on the linear mixing frequency extension of the dynamic factor model proposed by Mariano and Murasawa (2003) where quarterly variables growth rates (Δy_t^q) are described by the geometric mean of the unobserved monthly growth rates of the variable (Δx_t). In particular,

$$\Delta y_t^q = \frac{1}{3}\Delta x_t^m + \frac{2}{3}\Delta x_{t-1}^m + \Delta x_{t-2}^m + \frac{2}{3}\Delta x_{t-3}^m + \frac{1}{3}\Delta x_{t-4}^m \tag{6}$$

where Δy_t^q is observed every three months. As mentioned by Camacho et al. (2012), this approximation has the main shortcoming that even in the most simpler case, where equation (2) is replaced by $\Delta f_t = \mu_{s_t} + \eta_t$, quarterly growth rates will depend on current and lagged unobserved states $S_t, \ldots, S_{t-4}$. Therefore, when the model is wanted to be estimated by the approximate maximum-likelihood method a total of 2^5 different paths have to be considered at each t. Camacho et al. (2012) proposed to approximate the density of Δy_t^q by,

$$f\left(\Delta y_t^q\right) = \sum_{j=1}^{32} \pi_j^* f\left(\Delta y_t^q | s_t^* = j\right) \approx \sum_{i=1}^{2} \pi_i f\left(\Delta y_t^q | s_t = i\right)$$

where π_j^* (π_i) is the unconditional probability of being in the j-th (i-th) state out of the 32 (2) different states. Through a Monte Carlo study Camacho et al. (2012) showed their estimation procedure have small effects in terms of the quadratic probability score when the idiosyncratic variance of the quarterly indicator is high enough.

If this kind of approximation wants to be avoided, one possibility could be estimating the proposed model through the Bayesian inference approach of Kim and Nelson (1998). In this sense, Marcellino et al. (2013) has proposed a Bayesian estimation approach to the linear dynamic factor model of Mariano and Murasawa (2003) which could potentially be adapted to the non-linear framework here presented. Nevertheless, we propose a second and more simpler alternative which is base in a different approach to the mixing-frequency problem.

2.2.2 Stacking approach

In the present work, we adopt a different solution to the one followed by Camacho et al. (2012). Specifically, we rely on the "stacking approach" recently proposed by Blasques et al. (2014) to deal with mixing frequencies in the context of a linear dynamic factor model. These authors have shown that, under the state space representation, it is possible to deal with the mixing frequencies problem by stacking the higher frequency data (monthly) into a vector of observations which operates at the lower frequency (quarterly) while all the high frequency information is still preserved[3]. Here, we extend their model to account for non-linearities in the context of a DFMS model.

Following Blasques et al. (2014) notation, monthly variables x_τ^m (τ is the monthly time index) can be stacked into a quarterly observed vector (x_t^q) with quarterly time index t

[3]The "stacking approach" has the main advantage of avoiding the introduction of artificial missing values when quarterly variables are not observed (two months every quarter), which is needed in Mariano and Murasawa (2003) for estimation purposes

of the form,

$$x_t^q = \begin{pmatrix} x_{t,1}^q \\ x_{t,2}^q \\ x_{t,3}^q \end{pmatrix} = \begin{pmatrix} x_{3(t-1)+1}^m \\ x_{3(t-1)+2}^m \\ x_{3(t-1)+3}^m \end{pmatrix} \tag{7}$$

where $x_{t,i}^q$ is the i-th element of x_t^q, where t refers to the quarter the monthly observation belong to and i indicates the month within the t quarter. Therefore, $t = 1, \ldots, T$, $i = 1, 2, 3$ and $\tau = 1, \ldots, 3T$.

If we assume a monthly observable variable which dynamics is explained by the model described in equations (1)-(3) (for simplicity assume $\beta_i(L) = \beta_i$ and $\Phi_i(L) = 1$), the process can be described as,

$$\begin{aligned} \Delta x_\tau^m &= \beta_x \Delta f_\tau^m + \epsilon_\tau^m & (8)\\ \Phi_f(L)\left(\Delta f_\tau^m - \mu_{s_\tau}^m\right) &= \eta_\tau^m & (9)\end{aligned}$$

where $\epsilon_\tau^m \overset{iid}{\sim} N\left(0, \sigma_x^2\right)$ and $\eta_\tau \overset{iid}{\sim} N\left(0, \sigma_f^2\right)$. Adding a quarterly variable Δy_t which also depends on Δf_τ^m and using the stacked vector representation (7) for the unobserved common factor,
$\Delta f_t^q = \begin{pmatrix} \Delta f_{t,1}^q & \Delta f_{t,2}^q & \Delta f_{t,3}^q \end{pmatrix}'$, it is possible to write

$$\begin{aligned} \Delta y_t &= \beta_y \Delta f_{3t-2}^m + \beta_y \Delta f_{3t-1}^m + \beta_y \Delta f_{3t}^m + \xi_t \\ &= \begin{pmatrix} \beta_y & \beta_y & \beta_y \end{pmatrix} \Delta f_t^q + \xi_t \end{aligned} \tag{10}$$

where $\xi_t \overset{iid}{\sim} N\left(0, \sigma_\xi^2\right)$. Writing down the mixing-frequency DFMS model under the stacked representation, using simultaneously both a monthly and a quarterly variables, can be done through the state space representation of (8)-(10). Using Durbin and Koopman (2012) notation and assuming an AR(1) process for the factor for exposition reasons:

$$\begin{aligned} x_t &= Z\alpha_t + \epsilon_t & \epsilon_t &\sim N(0, H) \\ \alpha_{t+1} &= M_{s_t} + T\alpha_t + R\eta_t & \eta_t &\sim N(0, Q) \end{aligned} \tag{11}$$

where

$$\begin{aligned} x_t &= \begin{pmatrix} y_t & x_{t,1}^q & x_{t,2}^q & x_{t,3}^q \end{pmatrix}' & \alpha_t &= \begin{pmatrix} f_{t,1}^q & f_{t,2}^q & f_{t,3}^q \end{pmatrix}' \\ \epsilon_t &= \begin{pmatrix} \xi_t & \epsilon_{t,1}^m & \epsilon_{t,2}^m & \epsilon_{t,3}^m \end{pmatrix}' & \eta_t &= \begin{pmatrix} \eta_{t,1}^q & \eta_{t,2}^q & \eta_{t,3}^q \end{pmatrix}' \end{aligned}$$

$$\begin{aligned} Z &= \begin{pmatrix} \beta_y & \beta_y & \beta_y \\ \beta_x & 0 & 0 \\ 0 & \beta_x & 0 \\ 0 & 0 & \beta_x \end{pmatrix} & M_{s_t} &= \begin{pmatrix} (1-\phi_f L)\mu_{s_{t,1}}^q \\ (1-\phi_f^2 L^2)\mu_{s_{t,2}}^q \\ (1-\phi_f^3 L^3)\mu_{s_{t,3}}^q \end{pmatrix} \\ T &= \begin{pmatrix} 0 & 0 & \phi_f \\ 0 & 0 & \phi_f^2 \\ 0 & 0 & \phi_f^3 \end{pmatrix} & R &= \begin{pmatrix} 1 & 0 & 0 \\ \phi_f & 1 & 0 \\ \phi_f^2 & \phi_f & 1 \end{pmatrix} \end{aligned}$$

$Q = 1$ and H is a diagonal matrix which entries in the main diagonal are determined by the vector $(\sigma_\xi^2, \sigma_\epsilon^2, \sigma_\epsilon^2, \sigma_\epsilon^2)$. Assuming AR(p) process for the idiosyncratic part of observable variables or for the factor is straightforward[4].

[4]In Blasques et al. (2014) can be found a detailed explanation of how to write down matrices T, R and Q for AR(p) process when using the stacking approach

3 Estimation Procedure

The non-linear stacked model discussed above could be estimated, among other methodologies, by using the approximate maximum-likelihood estimation (imposing equation (9) to be $f_\tau = \mu_{s_\tau} + \eta_\tau$ we would be in a similar model to the one estimated by Camacho et al. (2012)) or through a Bayesian Markov chain Monte Carlo methodology similar to the one proposed by Kim and Nelson (1998). It should be remarked that, when the first estimation approach is selected, the stacked representation allows us to reduce the number of possible paths to be accounted for with respect to Camacho et al. (2012). This is the case since only three "monthly" lags of the factor appears at each t, making the number of possible paths to be equal to $2^3 = 8$ when a switching intercept is used instead of a switching mean (remind in Camacho et al. (2012) $2^5 = 32$ paths were possible). This reduction would make the estimation computationally feasible without the necessity of an additional approximation. Even though this possibility, the estimation procedure followed in this paper is based on Bayesian methods. Two main reasons are behind our selection. The first one is related with the observation done by Kim and Nelson (1998) regarding the difficulty of judging the effects of the approximate maximum-likelihood on the inference of parameters and unobserved components. In the case of Bayesian estimation, the exact filter is instead computed. The second motive is related with allowing the model to be estimated using either a switching intercept or a switching mean[5].

3.1 Bayesian Inference using a MCMC methodology

A Metropolis Hasting algorithm within Gibbs Sampling is used to estimate the model. The basic procedure is based on Kim and Nelson (1998) where three basic steps are carried on. First, we draw the unobserved common component $(\Delta f_1, \ldots, \Delta f_T)$ conditional on the unobserved states $(S_1, \ldots, S_T)$ and all parameters (θ). This can be achieved by using a simulation smoother algorithm as proposed by Carter and Kohn (1994) or by Durbin and Koopman (2002). In a second step, the unobserved states $(S_1, \ldots, S_T)$ are generated conditional on the unobserved common component and all parameters (θ) through the multimove Gibbs-Sampling estimation described in Kim and Nelson (1999). Finally, conditional on the common factor and unobserved states, equations from (8)-(10) are independent, allowing for a separate treatment of each other. The identification assumption for the model stands on assuming the variance of the common unobserved component (σ_f) to be equal to one. We refer the reader to Kim and Nelson (1999) to a further detailed explanation of the involved procedure.

Since mixing frequencies are used in our model, one main clarification should be done regarding the draw of the parameters associated with quarterly variables (β_y, σ_y). Assuming an AR(p) process for those variables, we have

$$\begin{aligned} \Delta y_t &= \begin{pmatrix} \beta_y & \beta_y & \beta_y \end{pmatrix} \Delta f_t^q + u_{y,t} \\ u_{y,t} &= \phi_{1,y} u_{y,t-1} + \ldots + \phi_{p,y} u_{y,t-p} + \xi_t \end{aligned} \tag{12}$$

Conditional on $(\phi_{1,y}, \ldots, \phi_{p,y})$ and $(\Delta f_1, \ldots, \Delta f_T)$, it is possible to pre-multiply both sides of equation (12) by $(\Phi_y = 1 - \phi_{1,y} - \ldots - \phi_{p,y})$ to obtain,

[5]Using a switching mean in the context of the approximate maximum-likelihood estimation means increasing the number of path to be taking into account to 2^{p_f+3}, where p_f refers to the order of the AR process selected for the factor

$$\Phi_y \Delta y_t = \begin{pmatrix} \beta_y & \beta_y & \beta_y \end{pmatrix} \begin{pmatrix} \Phi_y \Delta f^q_{t,1} \\ \Phi_y \Delta f^q_{t,2} \\ \Phi_y \Delta f^q_{t,3} \end{pmatrix} + \xi_t \quad (13)$$

Different from equation (12), equation (13) has uncorrelated residuals, allowing for the possibility of using a Normal-gamma conjugate prior for the estimation of β_y and σ_y. In this particular case, the introduction of the stacking representation allows us to simplify the estimation of the parameters associated with quarterly variables when compared to the methodology proposed by Marcellino et al. (2013). In their work, the approach of Mariano and Murasawa (2003) is instead used, meaning equation (12) of our specification turns into,

$$\begin{aligned} \Delta y_t = {} & \frac{1}{3}\beta_y \Delta f_t + \frac{2}{3}\beta_y \Delta f_{t-1} + \beta_y \Delta f_{t-2} + \frac{2}{3}\beta_y \Delta f_{t-3} + \frac{1}{3}\beta_y \Delta f_{t-4} + \\ & \frac{1}{3}u_{y,t} + \frac{2}{3}u_{y,t-1} + u_{y,t-2} + \frac{2}{3}u_{y,t-3} + \frac{1}{3}u_{y,t-4} \end{aligned} \quad (14)$$

As noticed by Marcellino et al. (2013), two main difficulties have to be sorted to estimate β_y. First, since time is monthly specified (in the stacked approach each t refers to quarters) there are two missing observations every quarter for quarterly variables. This is easily solved by using only true observations for the estimation. The second, and most important difficulty, is the appearance of a MA(4) regression error in equation (14), where the error (u_t) follows an AR(p) process. To overcome this problem Marcellino et al. (2013) proposed to work out the variance covariance matrix of the error term of equation (14), $\Theta(\Phi_y, \sigma_y)$, and pre-multiply both side of (14) by $\Theta^{-\frac{1}{2}}$ in order to obtain a standard regression with uncorrelated residuals.

4 Empirical Application

4.1 In-sample analysis: US case

The mixing frequency framework allows for the possibility of adding the real GDP measure to the four monthly coincident indicators commonly used when analyzing the US business cycles. In other words, five variables are included in our model: real gross domestic product (GDP), the industrial production index (IP), real personal income less transfer payments (INC), real manufacturing and trade industry sales (SLS) and employees on non-agricultural payrolls (EMP)[6]. When included in the model, all variables are demeaned after computed a hundred time their change in the natural logarithm. The data sample is from January 1959 to September 2014 and was obtained from the Federal Reserve Economic Data (FRED) of the Federal Reserve Bank of St. Louis.

An AR(2) process is assumed for both the unobserved common factor and for the idiosyncratic components of equation (1) and, as in Kim and Nelson (1998), we adopt the "compact" state-space representation –after adapting it to the stacking approach– in

[6]The series related with employment could instead be replaced by "employee-hours on non-agricultural payrolls" (Stock and Watson (1989)), or by "non-agricultural civilian employment" (Chauvet (1998) and Chauvet and Hamilton (2006)).

order to conduct our Bayesian estimation. This is achieved by pre-multiplying both sides of equation (1) by $(1 - \phi_1^i L + \phi_2^i L^2)$ in order to obtain the quasi-difference equation

$$y_{it}^* = \beta_i(L) f_{it}^* + \epsilon_{it} \tag{15}$$

where $y_{it}^* = \Delta y_{i,t} - \phi_1^i \Delta y_{i,t-1} - \phi_2^i \Delta y_{i,t-2}$ and $f_{it}^* = \Delta f_t - \phi_1^i \Delta f_{t-1} - \phi_2^i \Delta f_{t-2}$.

As noticed in Stock and Watson (1989), payroll employment could not be exactly a coincident but a lagging indicator. Therefore, we follow Stock and Watson (1989) and Kim and Nelson (1998) and we include three lags of f_t^* in the payroll equation. In Appendix A we describe the "compact" state-space representation for this particular setting.

Priors selected are quite diffuse. In particular, the prior for all factor loading coefficients is $N(0, 1000)$, for standards deviation of observable variables an inverted gamma distribution of the form $IG = (6; .0001)$ is chosen, while for the parameters of autoregressive polynomials is set equal to $N(0, \Sigma)$, where $\Sigma = [1 \;\; 0; 0 \;\; .5]$[7]. The prior for the regime switching means (μ_0 and μ_1) is $N(0, I_2)$[8]. Following Kim and Nelson (1998) informative priors are used for transition probabilities.

Starting values are set as follows: we generate values for f_t from $t = 1, \ldots, T$ from a $N \sim N(0, 1)$. Using those random values we run an OLS regression for each variable and we obtain values for the loading factors and the standard errors of the idiosycratic term of each variable. Therefore, while starting values for factor loadings values are close to zero, starting values for standard errors are around the unity. Starting values for autoregressive parameters are set to zero while for the switching means are -2 and 2.5 for the recession and expansion periods respectively.

Table 1 presents Bayesian posterior distributions of the parameters when the stacked approach is used. Two different alternatives are shown, the top of the table refers to the mixing frequency case while the bottom are results when no quarterly variable is included in the model (as in Kim and Nelson (1998)). From the table we can observe the inclusion of a quarterly variable among observable variables does not affect parameters estimation with respect to the non-mixing frequency case.

Figure 1 shows the in-sample-smoothed posterior probabilities of recessions for both, the mixing frequency (black line) and the non-mixing frequency case (red line). As usually, shaded areas are NBER recession periods. The only remarkable difference between both models is found with respect to the 1973's recession. In that particular case, the in-sample probabilities of the mixing frequency model shows stronger signals of an ongoing recession. Nevertheless, similar to Camacho et al. (2012), the figure indicates no large improvements are obtained when including the GDP measure. This is in line with results obtained in Camacho et al. (2014), where it is shown that improvements attained in terms of business cycle identification decrease with the number of the high quality indicators already used, even when the added indicator has a larger signal-to-noise ratio.

4.2 Estimating Turning Points in real-time

Hamilton (2011) has pointed out the importance that earlier identification of business cycle tuning points could have in terms of a promptly economic policy reaction. Analyzing

[7]As mentioned in Kose et al. (2003) the idea behind this kind of prior is that growth is not serially correlated. Following them, we also rely in Metropolis-Hasting step in order to ensure stationarity

[8]In order to ensure $\mu_1 > 0$, if the generated value is less than or equal to 0, the draw is discarded

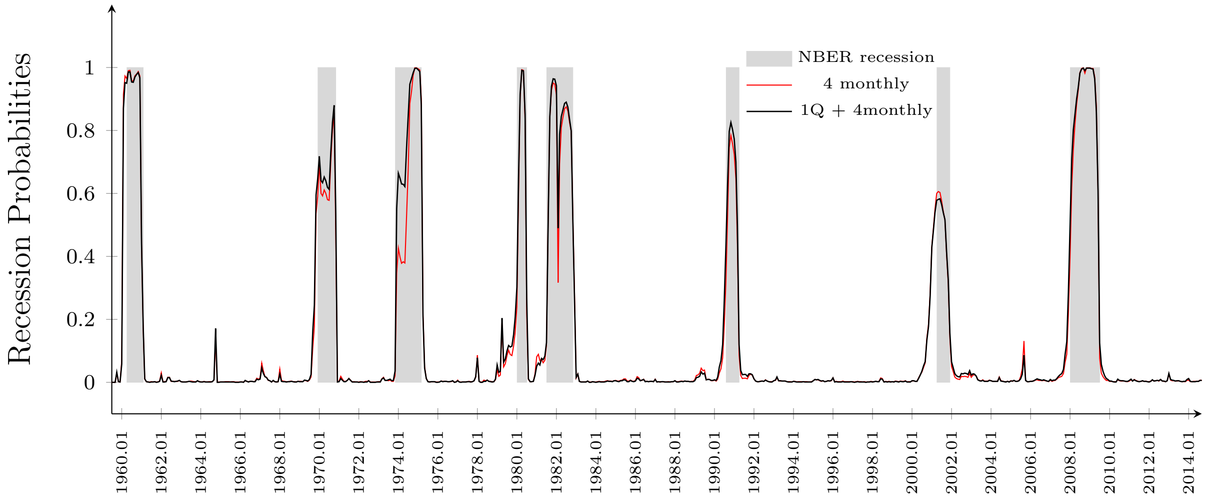
Table 1: Parameter estimates

	1 Quarterly and 4 monthly variables included					
Param	IP	INC	SLS	EMP	GDP	Factor
β	.485	.184	.412	.094	.197	
	[0.466; 0.503]	[0.173; 0.195]	[0.394; 0.429]	[0.089; 0.099]	[0.187; 0.207]	
β_2				.014		
				[0.009; 0.019]		
β_3				.014		
				[0.01; 0.018]		
β_4				.025		
				[0.021; 0.029]		
ϕ_1^i	$-.067$	$-.199$	$-.414$	.097	$-.218$	.224
	[-0.109; -0.027]	[-0.226; -0.173]	[-0.442; -0.384]	[0.068; 0.127]	[-0.269; -0.167]	[0.182; 0.272]
ϕ_2^i	$-.113$	$-.076$	$-.208$	.379	$-.044$	.188
	[-0.149; -0.077]	[-0.103; -0.049]	[-0.236; -0.179]	[0.349; 0.41]	[-0.097; 0.009]	[0.144; 0.235]
σ_i^2	.188	.274	.533	.015	.304	1.000
	[0.175; 0.201]	[0.263; 0.284]	[0.509; 0.555]	[0.014; 0.016]	[0.281; 0.325]	
	μ_0	μ_1	$\mu_0+\mu_1$	q	p	
	-1.934	2.213	0.279	0.871	0.981	
	[-2.128; -1.749]	[2.029; 2.415]	[0.219; 0.343]	[0.844; 0.906]	[0.976; 0.986]	

	4 monthly variables included					
Param	IP	INC	SLS	EMP	GDP	Factor
β	.501	.183	.421	.091	–	
	[0.482; 0.519]	[0.171; 0.194]	[0.403; 0.439]	[0.087; 0.096]	–	
β_2				.015		
				[0.01; 0.02]		
β_3				.015		
				[0.011; 0.019]		
β_4				.026		
				[0.022; 0.003]		
ϕ_1^i	$-.088$	$-.190$	$-.418$	.109	–	.219
	[-0.132; -0.046]	[-0.217; -0.162]	[-0.446; -0.388]	[0.08; 0.138]	–	[0.177; 0.266]
ϕ_2^i	$-.109$	$-.066$	$-.213$	.391	–	.157
	[-0.147; -0.071]	[-0.094; -0.04]	[-0.241; -0.184]	[0.362; 0.421]	–	[0.112; 0.204]
σ_i^2	.170	.277	.527	.016	–	1.000
	[0.156; 0.183]	[0.266; 0.287]	[0.504; 0.549]	[0.015; 0.016]	–	
	μ_0	μ_1	$\mu_0+\mu_1$	q	p	
	-1.971	2.24	0.269	0.861	0.98	
	[-2.161; -1.784]	[2.058; 2.435]	[0.21; 0.331]	[0.832; 0.898]	[0.976; 0.986]	

Note: 4 monthly variables refers to a similar model to the one estimated in Kim & Nelson (1998). β are loading factors, ϕ stands for autoregressive parameters, μ_0 and μ_1 are recession and expansion means, q and p are recession and expansion transition probabilities. Values showed are the mean, the standard deviations (within parenthesis) and the 75% bands (within brackets) from the posterior distribution. In both cases, the first 10000 draws in the Gibbs simulation were discarded, while the next 40000 draws were used for the estimation.

Figure 1: In-sample US smoothed recession probabilities. Mixing vs. non-mixing frequency models



Note: Both alternatives are computed using 40000 draws in the Gibss simulation (after discarding the first 10000 draws.

the performance of a model from that perspective calls for the use of real-time data sets. Following this direction, we evaluate our multivariate mixing-frequency DFMS model as if it would have been used to estimate US recession probabilities during the last 35 years. Our data sets includes data vintages from January 1977 to August 2014 for all five variables included in the model (depending on the variable the sample starts in January 1959 or in January 1967)[9].

One important analysis to be done when real-time analysis is carried on is when the information for each variables is released. As it is known, in the case of US' economic variables, advance data announcements for each monthly or quarterly variable ("advance estimate") are followed by data revisions during two consecutive months ("second" and "third" estimates). After the "third" estimate release more revisions could be done which, in general, affects the whole series (annual revisions, change of base period, etc.). In the case of monthly variables, "advance" estimates of month τ for EMP, IP and INC are published in month $\tau+1$ while "advance" information for SLS is released in month $\tau+2$. For GDP, "advance" estimates for quarter $t = \lfloor \tau/3 \rfloor$ are published at the end of the month following the end of the t quarter. Variables' data releases are never done all at the same day of the month, meaning the available amount of information varies within a month. We follow Chauvet and Piger (2008) and we assume estimation for each vintage is done at the last day of the month. This implies that at any month τ, non-revised available monthly data for EMP, IP and INC is up to month $\tau-1$, while SLS is available up to $\tau-2$. Regarding GDP, information would be available up to quarter $t-1 = \lfloor (\tau-3)/3 \rfloor$ and depending whether it is the first, the second or the third month of the quarter, information would respectively be the "advance", the "second" or the "third" release[10].

[9]Vintages for GDP, IP and EMP were obtained from the Federal Reserve Bank of Philadelphia real-time data archive. We are grateful to Gabriel Perez-Quiros for kindly sharing the data vintages of SLS and INC.

[10]Camacho et al. (2012) assume to be stand in the middle of the month meaning at month τ available information for IP and EMP is up to month $\tau-1$, INC would be available up to $\tau-2$ and SLS up to

Not only timing of data release should be taking into account when working with real-time data, but also which data release is better to be used. When estimating business cycles' turning points using only GDP data, Chauvet and Hamilton (2006) recommend waiting after the second estimate of the previous quarter is released to start using information about two quarters ago. As an example, this would mean waiting until August to make inference about the first quarter of the current year. This recommendation is done in the context of a univariate Markov switching model (see Hamilton (1989)) and it is based on the fact that GDP data revisions could be of a significant magnitudes. Also aiming to estimate business cycle turning points but base only on monthly data, Chauvet and Hamilton (2006) and Chauvet and Piger (2008) suggest using the second estimate of EMP, IP and INC and the advance data release of SLS. This decision is basically based on the necessity of a balanced panel data set for estimation purposes. However, as noticed in Camacho et al. (2012), ragged-ends (or unbalanced panel data set) could easily be incorporated in the context of a multivariate DFMS model where state-space models and the Kalman filter are used[11].These authors have shown, both theoretically and through a Monte Carlo experiment, that when data for different variables are released with dissimilar lags, using an unbalanced panel data with the latest available information helps to improve the inference about the current state of the cycle.

Having into account these considerations, we analyze the business cycles dating performance of our mixing frequency DFMS model both, using only revised data for all variables but SLS or using advanced releases for all variables (latest available information). The first case is in order to compare our framework with the non-mixing frequency DFMS model used in Chauvet and Piger (2008) (excluding GDP from our model we are under their specific setting). The second approach, helps us to empirically analyze results of Camacho et al. (2012) related with the existing trade-off between using latest available data (potentially sooner estimation of turning points but unrevised data) or using a "better quality" data (potentially more accurate estimation of turning points but delay in identifying those).

In order to identify a business cycle turning point some ad-hoc assumptions must be done regarding how to declare a recession period has started (or finished). In this sense, we rest on the two step dating decision rule of Chauvet and Piger (2008). Their rule indicates at the first step that every time recession probability values move from below (above) to above (below) 80% (20%) and remain at least three consecutive periods above (below) that value, a recession (expansion) period is taking place. In the second step, the first month of each recession (expansion) period is identified as the month –prior to the first month with a 80% (20%) value– where recession probabilities move from below (above) to above (below) 50%. Based on this decision rule, we estimate US recession probabilities and identify US business cycles' turning points for the period April 1959 to December 1976. We then recursively increased the time-span by one month and we re-estimated the model generating real-time recession probabilities for each month from January 1977 to August 2014. One clarification must be done when using the stacked representation for real-time estimation. For each particular vintage analyzed, zero, one or two months of missing observations need to be included at the end of the sample. In other words, assuming we want to compute recession probabilities with data up to

$\tau - 3$. In the case of GDP, it means to exclude last available release.

[11]Our strategy to deal with unbalanced panel data rely on skipping missing observations for the updating Kalman filter equations of the simulation smoother algorithm.

January 2014, our model forces us to incorporate missing observations for the months of February and March 2014. Nevertheless, as it was said before, missing observations can be easily handle in this context.

In Table 2 it is shown business cycles' turning points as declared by the NBER and those identified by both Chauvet and Piger (2008)'s model and by our mixing-frequency DFMS model. As it was said, two cases are taking into account: using revised data and using the latest available information. Given two of our series (SLS and INC) starts in 1967 and not in 1959 as in Chauvet and Piger (2008), plus we use the stacking representation and that we introduce some minor differences regarding estimation assumptions[12], we decide to re-estimate turning points under the "stacking" non-mixing frequency DFMS model. Some discrepancies are found between their and our results (columns 2 vs. 5 of the table). In particular, it can be observed the 2000/01 peak date is identified for us in November 2000 while Chauvet and Piger (2008) pointed out January 2001. In the case of through dates, there is one month difference for the 2008 financial crisis (June 2009 in our estimations and July 2009 in their case). Given these differences, a fair comparison between models should be done using "re-estimated" turning points for the non-mixing frequency DFMS.

Following the literature, we assume NBER's peak and through dates as being the true ones. From January 1977 to August 2014 five peaks and five through are pointed out by the NBER. Each of the alternative estimations identified all 10 turning points and none of them give a "false" positive. The first comparison to be done is related with the turning points we get by using "advance" data or when using only "second" estimations. When analyzing the non-mixing frequency model we observe only one peak (2000/2001) dates differ, while in the mixed-frequency case just the 1982 through dates differ (in both cases by one month). This result could mean differences between "advance" and "revised" does not affect the identification of a turning point. It should be reminded that, in the context of a univariate Markov switching model, Chauvet and Hamilton (2006) recommend waiting until the "second" GDP estimate of the previous quarter is released in order to use GDP data from to quarters ago for accuracy purposes. When analyzing whether there exist accuracy gains from the introduction of a quarterly variable in the model, we observe some evidence that this is the case. When "revised" data is used (column 3 vs. 5 of the table) we see for the 1981, 2000/2001 and 2007/2008 peak dates the mixing-frequency model does better in getting closer dates to the ones pointed out by the NBER (3 out of 5 cases) and does never do worse. For the through dates, any improvement is found when using the mixing frequency DFMS model. Something similar occurs when we compare both non-revised data estimated models (column 4 vs. 6). In that case two peak dates (1981 and 2000/2001) and one trough date (1982) are better estimated in the mixing-frequency DFMS model. Altogether, it appears that enlarging the model by using GDP gives some gains when the target is to obtain the NBER turning points, with more impact on the peak's date identification. Moreover, the enlarged model never do worse than the non-mixing frequency case. Improvements in accuracy from adding a quarterly variable are also found by Camacho et al. (2012), but in their case improvements were higher when estimating the phase out of the crisis (throughts) than the peaks dates.

[12]We assume looser priors for the factor loadings and the autoregresive parameters. A rejection algorithm is used for ensure stationarity in Chauvet and Piger (2008) while a Metropolis-Hasting step is used in our model.

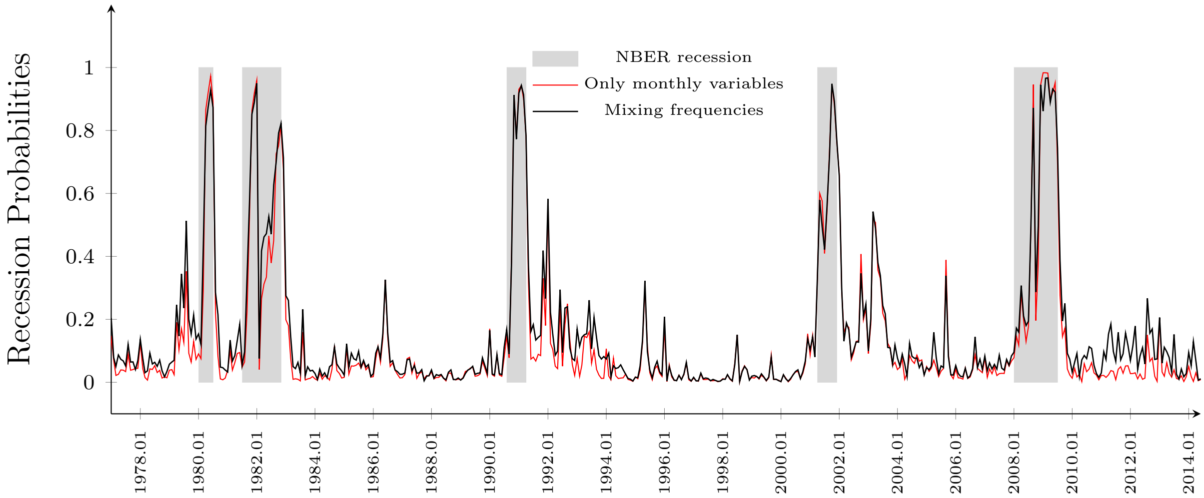
Table 2: Business cycle turning points dates: NBER and estimations

RECESSIONS											
PEAK DATE AS DETERMINED BY:						PEAK DECLARATION DATE AS ANNOUNCED BY:					
NBER	Chauvet & Piger (2008)	Mixing Frequencies (revised data)	Mixing Frequencies	Non-Mixing Frequencies (revised data)	Non-Mixing Frequencies	NBER	Chauvet & Piger (2008)	Mixing Frequencies (revised data)	Mixing Frequencies	Non-Mixing Frequencies (revised data)	Non-Mixing Frequencies
Jan 1980(Q2)	Jan 1980	Jan 1980	Jan 1980	Jan 1980	Jan 1980	Jun 3, 1980	Jul 1980	Jul 1980	Jun 1980	Jul 1980	Jun 1980
Jul 1981(Q3)	Aug 1981	Jul 1981	Jul 1981	Aug 1981	Aug 1981	Jan 6, 1982	Feb 1982	Feb 1982	Jan 1982	Feb 1982	Jan 1982
Jul 1990(Q3)	Jul 1990	Jul 1990	Jul 1990	Jul 1990	Jul 1990	Apr 25, 1991	Feb 1991	Feb 1991	Feb 1991	Feb 1991	Feb 1991
Mar 2001(Q1)	Jan 2001	Dec 2000	Dec 2000	Nov 2000	Dec 2000	Nov 26, 2001	Jan 2002	Dec 2001	Nov 2001	Dec 2001	Nov 2001
Dec 2007(Q4)	Feb 2008*	Jan 2008	Jan 2008	Feb 2008	Feb 2008	Dec 1, 2008	Jan 2009*	Nov 2008	Oct 2008	Nov 2008	Oct 2008

EXPANSION											
TROUGH DATE AS DETERMINED BY:						TROUGH DECLARATION DATE AS ANNOUNCED BY:					
NBER	Chauvet & Piger (2008)	Mixing Frequencies (revised data)	Mixing Frequencies	Non-Mixing Frequencies (revised data)	Non-Mixing Frequencies	NBER	Chauvet & Piger (2008)	Mixing Frequencies (revised data)	Mixing Frequencies	Non-Mixing Frequencies (revised data)	Non-Mixing Frequencies
Jul 1980(Q3)	Jun 1980	Jun 1980	Jun 1980	Jun 1980	Jun 1980	Jul 8, 1981	Dec 1980	Dec 1980	Nov 1980	Dec 1980	Nov 1980
Nov 1982(Q4)	Oct 1982	Oct 1982	Nov 1982	Oct 1982	Oct 1982	Jan 8, 1983	May 1983	May 1983	Apr 1983	May 1983	Apr 1983
Mar 1991(Q1)	Mar 1991	Mar 1991	Mar 1991	Mar 1991	Mar 1991	Dec 22, 1992	Sep 1991	Aug 1991	Aug 1991	Aug 1991	Jul 1991
Nov 2001(Q4)	Nov 2001	Nov 2001	Nov 2001	Nov 2001	Nov 2001	Jul 17, 2003	Aug 2002	Aug 2002	Jun 2002	Aug 2002	Jul 2002
Jun 2009(Q2)	Jul 2009*	Jun 2009	Jun 2009	Jun 2009	Jun 2009	Sep 20, 2010	Jan 2010	Jan 2010	Dec 2009	Nov 2009	Nov 2009

Note: * Values taken from Hamilton (2011).

Figure 2: Real-time US smoothed recession probabilities. Mixing vs. non-mixing frequencies DFMS models



Note: Both alternatives were computed at each t using 8000 draws in the Gibss simulation (after discarding the first 2000 draws). Latest available information is always used.

Table 2 also shows months when each of the models would have announced a turning point according to Chauvet and Piger (2008) dating rule. When the focus is on peak dates, it can be observed that enlarging the model by including the GDP among the variables do not change the announcement date between the mixing and the non-mixing frequency models. However, in line with Camacho et al. (2012), we observed that in all cases but the 1990 recession, peak dates are announced one month in advance when latest available information is included. This result comes from exploiting the possibility, proposed by Camacho et al. (2012), of dealing with unbalanced panel data in the MSDF model. In our setting, this means having one month of additional information for three out of the four monthly variables included. In the case of through announcements dates, adding GDP data means delaying declaration date by two months for the last recession when only revised data is used (column 9 vs. 11) while the announcements dates differ for the last three thoughts when results of using last available information are compared (column 10 vs. 12). Regarding the use of latest available information, as in the peak case, we found that through announcement dates would have been done (in general) one month in advance. When comparing announcement dates with the NBER declaration dates, the use of the latest available information implies the model would have declare peaks date during the same month as the NBER in three cases and two months in advance in the other two. In declaring the end of a recession using unbalanced panel data means earlier announcements than the NBER in all cases but the 1981/1982 recession.

In line with Camacho et al. (2012), results points out the necessity of using the latest available information for promptly identification of business cycle turning points. Regarding using mixing frequencies, we don't find gains in terms of how quickly the announcement of a new phase of the business cycle has started. However, there are gains on using this kind of models when the aim is on a more accurate identification of the turning point (thinking on the NBER turning points as the true one).

In Figure 2 we plot real-time recession probabilities for the US using latest available information for the non-mixing and the mixing-frequency case. As it can be observed, there is no big difference between both models around NBER recession periods. This result diverges from Camacho et al. (2012) who found differences between the two cases (see Figures 7 and 8 from their working paper).

4.3 Forecasting Output

Predicting the evolution of the GDP is also of a main importance from a decision making point of view. Not only when the focus is on the implementation of long-term policies, but also in the very short-term for the day-to-day decisions. In this sense, the development of models aiming to obtain promptness accurate predictions about output's evolution (now-casting or backcasting estimations) has been growing during the last decades. Reviewing a large amount of literature on this field, Chauvet and Potter (2012) point out factor models, non-linearities and mixing-frequencies as relevant issues when writing down the model to be used in predicting the short-term evolution of the GDP. Using real-time data for the US economy, the authors compare 13 different models in terms of forecast accuracy (measured in terms of both the root mean square error –RSME– and the Theil inequality coefficient), and found differences in predicting output during recession and expansion phases. In fact, those models which do reasonable good during expansions (but not significantly better than a simple AR(2) model) are not those which do good enough during recessions. When the attention is set on downturn periods, Chauvet and Potter (2012) found that augmenting the simple AR(2) model by including as regressors the unobserved common component and the switching-states probabilities obtained from a multivariate DFMS model, it is possible to get significantly better forecasting accuracy compared to the simple AR(2) case and all the other models there studied [13]. However, in order to estimate the "augmented" model they need to use a "two step" procedure since GDP measures are absent among the variables included in the DFMS. The stacking approach presented above gives us the opportunity of directly analyze the forecast accuracy for the GDP when a mixed frequencies DFMS is used.

Aiming to measure the predictive accuracy of our model, we compare the nowcast performance of our model relative to the simple AR(2) and to the one proposed by Chauvet and Potter (2012). The analysis also take into account the specific month of the quarter when the predictive value is estimated. To obtain the GDP nowcast values we basically rely on the state-space representation of section 2.2.2 which is not the same as the "compact" state-space representation[14]. In this sense, we first estimate all the parameters and the unobserved states values using the methodology explained in section 3 to fill-in the new needed matrices. Relying on the Kalman Filter, we then estimate the common unobserved component ($\triangle f_t$) up to the $t+h$ period (where h is the h-period ahead to be forecast). The predicted value for $\triangle y_{i,t+h}$ is then easily obtained. It should be noticed that if we had used the "compact" state space representation instead, the forecast would have been about $y^*_{i,t+h} = \triangle y_{i,t+h} - \phi^i_1 \triangle y_{i,t+h-1} - \phi^i_2 \triangle y_{i,t+h-2}$ and not for $\triangle y_{i,t+h}$.

Compare to Chauvet and Potter (2012), we extend the period where forecast accuracy is analyzed while more information is added for each particular vintage. In particular, Chauvet and Potter (2012) study the performance of their model using the last balanced

[13] Camacho et al. (2013) found gains in using mixing-frequency dinamic factor models when forecast the European cycle. However, they do not find evidence of a better forecast accuracy when non-linear models are used

[14] see Appendix B for matrices' representation

data-vintages that can be obtained at the end of March, June, September and December of each year. Regarding quarterly data, this means using the "third estimate" of the GDP for each quarter[15], while monthly information correspond to data up to $t-2$, meaning monthly data will be up to January, April, July and October depending on the vintage analyzed. In other words, the latest available data for the IP, EMP and INC is not used (those three variables have available information up to $t-1$ in each particular vintage t). This is not the case when estimating the mixing frequency DFMS model here proposed, since unbalanced panel data sets can be used. Moreover, we distinguish at which particular vintage of each quarter the nowcast exercise is conducted. In other words, depending whether vintage t is the first, the second or the third month of the current quarter different values will be obtained. Differences comes from the amount of monthly available information at each particular month and whether the "advance", the "second" or the "third" estimates values for GDP of the previous quarter is used. In all three cases the target would be the official "third" GDP estimate published three months after the end of the current quarter. In other words, nowcasts are done between 3 and 5 months prior to the publication of our target. Regarding the sample period extension we have estimated the model using data from 1959Q1 to 1976Q4 and we have re-estimated it recursively generating noecasts for each quarter from 1977Q1 to 2013Q4. As in Chauvet and Potter (2012) we compare nowcast accuracy related to the simple AR(2) model in terms of RSME and the Theil inequality values. To study when the nowcast accuracy are significantly different we rely in both the nested tests proposed by Clark and McCracken (2005) for the augmented AR(2) model and the Diebold and Mariano (2002) test for the MF-DFMS model.

In Table 3 we present the RMSE and the Theil inequality for the output nowcast for the three models under analysis. The mixing frequency DFMS model improves the nowcasts accuracy –with respect to the benchmark AR(2) model– when some of the monthly information about the current quarter has already been released. In other words, using the mixing frequency DFMS model during the second or the third month of the current quarter significantly increase the GDP's nowcast accuracy with respect to the AR(2) model. The improvement occurs regardless the nowcast is done during expansion or recession phases, even though the difference is not significant during expansions in case of Panel B (estimations done during the 2^{nd} of the quarter). When the nowcast is done during the first month of the quarter and no information about the current quarter is published yet, no significant gains are obtained with respect to the simple AR(2) model. Nevertheless, a lower RSME value is obtained for recessions periods.

Regarding the Theil inequality coefficient, results show in all cases the mixing frequency DFMS model does it relatively better than the AR(2) model. Improvements are also found when the focus is on tracking the variance of the GDP (Theil Inequaility Variance statistic).

To compare values obtained with the augmented AR(2) model to the ones obtained with the mixed frequency DFMS model, we only focus on nowcast done during the 3^{rd} month of the quarter as Chauvet and Potter (2012) propose. Comparing values obtained during the 1^{st} or the 2^{nd} month of the quarter wouldn't have been complete fair, since no monthly information about the current quarter would have been use in their case. As it can be observed at Panel A of table 3 values relative to the simple AR(2) model are

[15] As an example, the 1990 March vintage correspond to the "third estimate" of the last quarter of 1989

Table 3: RMSE and Theil Inequality
(1977.Q1 to 2013.Q4 vintages)

Panel A: Nowcast done during the 3^{rd} month of each quarter

Models	RSME			Theil Inequality			
	Full Sample	Expansion	Recession	Total	Bias	Var	Cov
AR(2)	2.634	2.121	4.536	.387	.014	.418	.574
Relative	1.00	1.00	1.00	1.00			
AR(2) - DFMS	2.198	1.788	3.747	.313	.016	.259	.732
Relative	.834***	.843***	.826**	.809			
MF - DFMS	1.900	1.801	2.365	.261	.041	.169	.797
Relative	.722***	.849**	.521**	.674			

Panel B: Nowcast done during the 2^{nd} month of each quarter

Models	RSME			Theil Inequality			
	Full Sample	Expansion	Recession	Total	Bias	Var	Cov
AR(2)	2.627	2.131	4.483	.386	.014	.425	.567
Relative	1.00	1.00	1.00	1.00			
MF - DFMS	2.204	1.943	3.287	.297	.086	.184	.733
Relative	.839**	.911	.733*	.768			

Panel C: Nowcast done during the 1^{st} month of each quarter

Models	RSME			Theil Inequality			
	Full Sample	Expansion	Recession	Total	Bias	Var	Cov
AR(2)	2.626	2.117	4.519	.389	.011	.442	.554
Relative	1.00	1.00	1.00	1.00			
MF - DFMS	2.661	2.271	4.208	.352	.108	.167	.731
Relative	1.014	1.073	.931	.906			

Note: AR(2)-DFMS refers to the augmented AR(2) model proposed by Chauvet and Potter (2012). MF-DFMS refers to the mixing-frequency DFMS model. (*), (**) and (***) refers to a 10%, 5% and a 1% statistically significant difference relative to the AR(2) model. In the case of the AR(2)-DFMS we use Clark and McCracken (2005) test for nested models. For the MF-DFMS model we employ the corrected Diebold and Mariano (2002) test.

smaller for the mixed frequency DFMS model when the full sample, recession periods or the Theil inequality are analyzed.

Results are in line with those shown for the comparison with Chauvet and Potter (2012).

5 Conclusions

An alternative and easily handle Bayesian approach for computing mixed frequencies multivariate dynamic factor models –both in the linear and the non-linear cases– has been proposed in the present paper. Different from most of the literature –which has generally stood on Mariano and Murasawa (2003) strategy for dealing with mixing frequencies– we have followed the stacking approach proposed by Blasques et al. (2014). This alternative strategy has given us the opportunity of solving some of the problems that has emerged when a Bayesian estimation has been conducted or when a non-linear extension of the model was proposed. To our knowledge, Marcellino et al. (2013) is the only existing attempt of Bayesian estimation of a linear mixed frequency multivariate dynamic factor model. There, the Bayesian estimation of coefficients related with low frequency observables (e.g. quarterly variables) require the approximation of variance-covariance matrices of MA(4) regressions equations where the errors follows –in turn– an AR(p) process. As it has been shown, our estimation strategy completely skip this problem. Regarding the estimation of non-linear mixed frequency multivariate dynamic factor models, both Camacho et al. (2012) and Camacho et al. (2014) are the main references in the literature. With respect to their work, we have shown our estimation strategy does not confront the "curse of dimensionality problem" where even in the most simpler case 32 parallel Kalman filters have to be evaluated. This allows us avoiding to approximate 32 (or even more) different paths using only 2 different states as in Camacho et al. (2012) and Camacho et al. (2014).

We have applied the new proposed methodology to estimate the multivariate DFMS model of Kim and Nelson (1998) but extending it to the mixed frequency case. In particular, we add the real GDP measure to the four monthly coincident indicator commonly used to track the U.S. business cycle. Results show that including output measures means a better accuracy in terms of a real-time distinction of the turning points of the US business cycle –when NBER recession dates are assumed as being the true ones–. When the focus is on the timing of the announcement a recession has started or ended, we have shown the possibility of using the latest available information –the model deals with unbalanced panel data– would have meant anticipating the NBER announcement dates for most of the recessions declared since 1980. Finally, compared with other existing methodologies it is has been shown our non-linear model has a better nowcast accuracy when at least one month of monthly data from the current quarter is already released (i.e. during the second and third month of the quarter).

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A Appendix: state space representations

Under the specific setting described in section 4.1, the "compact" state representation of the model can be written as

$$\begin{aligned} Y_t &= Z \times F_t + \epsilon_t & \epsilon_t &\sim N(0, H) \\ F_{t+1} &= M_{s_t} + T \times F_t + R\eta_t & \eta_t &\sim N(0, Q) \end{aligned} \tag{16}$$

where

$$Y_t' = (y_t^*, x_{t,1}^{1^*}, x_{t,2}^{1^*}, x_{t,3}^{1^*}, x_{t,1}^{2^*}, x_{t,2}^{2^*}, x_{t,3}^{2^*}, x_{t,1}^{3^*}, x_{t,2}^{3^*}, x_{t,3}^{3^*}, x_{t,1}^{4^*}, x_{t,2}^{4^*}, x_{t,3}^{4^*})$$

$$Z = \begin{pmatrix} \psi_2^y & \psi_2^y & \psi_2^y & \psi_1^y & \psi_1^y & \psi_1^y & \beta_y & \beta_y & \beta_y \\ 0 & 0 & 0 & 0 & \psi_2^1 & \psi_1^1 & \beta_x^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_2^1 & \psi_1^1 & \beta_x^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_2^1 & \psi_1^1 & \beta_x^1 \\ 0 & 0 & 0 & 0 & \psi_2^2 & \psi_1^2 & \beta_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_2^2 & \psi_1^2 & \beta_x^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_2^2 & \psi_1^2 & \beta_x^2 \\ 0 & 0 & 0 & 0 & \psi_2^3 & \psi_1^3 & \beta_x^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_2^3 & \psi_1^3 & \beta_x^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_2^3 & \psi_1^3 & \beta_x^3 \\ 0 & \psi_{45}^4 & \psi_{44}^4 & \psi_{43}^4 & \psi_{42}^4 & \psi_{41}^4 & \beta_x^{40} & 0 & 0 \\ 0 & 0 & \psi_{45}^4 & \psi_{44}^4 & \psi_{43}^4 & \psi_{42}^4 & \psi_{41}^4 & \beta_x^{40} & 0 \\ 0 & 0 & 0 & \psi_{45}^4 & \psi_{44}^4 & \psi_{43}^4 & \psi_{42}^4 & \psi_{41}^4 & \beta_x^{40} \end{pmatrix} \quad F_t = \begin{pmatrix} f_{t-2,1}^q \\ f_{t-2,2}^q \\ f_{t-2,3}^q \\ f_{t-1,1}^q \\ f_{t-1,2}^q \\ f_{t-1,3}^q \\ f_{t,1}^q \\ f_{t,2}^q \\ f_{t,3}^q \end{pmatrix} \quad \epsilon_t = \begin{pmatrix} \xi_t \\ \epsilon_{t,1}^1 \\ \epsilon_{t,2}^1 \\ \epsilon_{t,3}^1 \\ \epsilon_{t,1}^2 \\ \epsilon_{t,2}^2 \\ \epsilon_{t,3}^2 \\ \epsilon_{t,1}^3 \\ \epsilon_{t,2}^3 \\ \epsilon_{t,3}^3 \\ \epsilon_{t,1}^4 \\ \epsilon_{t,2}^4 \\ \epsilon_{t,3}^4 \end{pmatrix}$$

and $\psi_j^y = -\phi_j^y \beta_y$; $\psi_j^i = -\phi_j^i \beta_x^i$; $j = 1, 2$; $i = 1, 2, 3$ and $\psi_{41}^4 = -\phi_1^4 \beta_x^{40} + \beta_x^{41}$; $\psi_{42}^4 = -\phi_2^4 \beta_x^{40} - \phi_1^4 \beta_x^{41} + \beta_x^{42}$; $\psi_{43}^4 = -\phi_2^4 \beta_x^{41} - \phi_1^4 \beta_x^{42} + \beta_x^{43}$; $\psi_{44}^4 = -\phi_2^4 \beta_x^{42} - \phi_1^4 \beta_x^{43}$; $\psi_{45}^4 = -\phi_2^4 \beta_x^{43}$.

The variance-covariance matrices are given by $Q = 1$ and

$$H = diag(\sigma_\xi^2, \sigma_{\epsilon_1}^2, \sigma_{\epsilon_1}^2, \sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \sigma_{\epsilon_2}^2, \sigma_{\epsilon_2}^2, \sigma_{\epsilon_3}^2, \sigma_{\epsilon_3}^2, \sigma_{\epsilon_3}^2, \sigma_{\epsilon_4}^2, \sigma_{\epsilon_4}^2, \sigma_{\epsilon_4}^2)$$

Vectors and matrices for the transition equations are

$$M_{S_t}' = (0_{(6,1)}, \varphi_{s_{t,1}}, \varphi_{s_{t,2}}, \varphi_{s_{t,3}}) \qquad \eta_t' = (0_{(6,1)}, \eta_{t,1}^q, \eta_{t,2}^q, \eta_{t,3}^q)$$

$$T = \begin{pmatrix} 0_{(9,6)} & \begin{matrix} 0_{(3,3)} \\ I_3 \\ \begin{matrix} 0 & \phi_{f2} & \phi_{f1} \\ 0 & \phi_{f1}\phi_{f2} & \phi_{f1}^2 + \phi_{f2} \\ 0 & \phi_{f1}^2\phi_{f2} + \phi_{f2}^2 & \phi_{f1}^3 + 2\phi_{f1}\phi_{f2} \end{matrix} \end{matrix} \end{pmatrix} \qquad R = \begin{pmatrix} 0_{(9,6)} & \begin{matrix} 0_{(6,3)} \\ \begin{matrix} 1 & 0 & 0 \\ \phi_{f1} & 1 & 0 \\ \phi_{f1}^2 + \phi_{f2} & \phi_{f1} & 1 \end{matrix} \end{matrix} \end{pmatrix}$$

The values for the vector containing the switching means comes directly from expanding equation (2) when the stacking approach is used:

$$\begin{aligned} \varphi_{s_{t,1}} &= \mu_{S_{t,1}}^q - \phi_{f1}\mu_{S_{t-1,3}}^q - \phi_{f2}\mu_{S_{t-1,2}}^q \\ \varphi_{s_{t,2}} &= \mu_{S_{t,2}}^q - (\phi_{f1}^2 + \phi_{f2})\mu_{S_{t-1,3}}^q - (\phi_{f1}\phi_{f2})\mu_{S_{t-1,2}}^q \\ \varphi_{s_{t,3}} &= \mu_{S_{t,3}}^q - (\phi_{f1}^3 + 2\phi_{f1}\phi_{f2})\mu_{S_{t-1,3}}^q - (\phi_{f1}^2\phi_{f2} + \phi_{f2}^2)\mu_{S_{t-1,2}}^q \end{aligned}$$

B

Under the specific setting described in section 4.1, the large state representation used for the nowcasting exercise can be written as

$$
\begin{aligned}
Y_t &= Z \times F_t + \epsilon_t & \qquad \epsilon_t &\sim N(0, H) \\
F_{t+1} &= M_{s_t} + T \times F_t + R\eta_t & \qquad \eta_t &\sim N(0, Q)
\end{aligned}
\tag{17}
$$

where

$$
Y_t' = (\Delta y_t, \Delta x^1_{t,1}, \Delta x^1_{t,2}, \Delta x^1_{t,3}, \Delta x^2_{t,1}, \Delta x^2_{t,2}, \Delta x^2_{t,3}, \Delta x^3_{t,1}, \Delta x^3_{t,2}, \Delta x^3_{t,3}, \Delta x^4_{t,1}, \Delta x^4_{t,2}, \Delta x^4_{t,3})
$$

$$
Z = \begin{pmatrix}
0 & 0 & 0 & \beta_y & \beta_y & \beta_y & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 & & & \beta_x^1 & 0 & 0 & 0 & 0 & 0 & & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & 0 & \beta_x^1 & 0 & \vdots & \vdots & \vdots & I_3 & \vdots & \ddots & & \ddots & & \ddots & \vdots \\
 & & & 0 & 0 & \beta_x^1 & & & 0 & & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \beta_x^2 & 0 & 0 & \vdots & \vdots & 0 & \cdots & 0 & & 0 & \cdots & 0 & \cdots & 0 \\
 & & & 0 & \beta_x^2 & 0 & & & \vdots & \ddots & \vdots & I_3 & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & 0 & 0 & \beta_x^2 & \vdots & \vdots & 0 & \cdots & 0 & & 0 & \cdots & 0 & \cdots & 0 \\
 & & & \beta_x^3 & 0 & 0 & & & 0 & \cdots & 0 & \cdots & 0 & & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & 0 & \beta_x^3 & 0 & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & I_3 & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \beta_x^3 & & & 0 & \cdots & 0 & \cdots & 0 & & 0 & \cdots & 0 \\
\beta_x^{43} & \beta_x^{42} & \beta_x^{41} & \beta_x^{40} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & & \\
0 & \beta_x^{43} & \beta_x^{42} & \beta_x^{41} & \beta_x^{40} & 0 & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & I_3 & \\
0 & 0 & \beta_x^{43} & \beta_x^{42} & \beta_x^{41} & \beta_x^{40} & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & &
\end{pmatrix}
$$

$$
\begin{aligned}
F_t' = (&\Delta f^q_{t-1,1}, \Delta f^q_{t-1,2}, \Delta f^q_{t-1,3}, \Delta f^q_{t,1}, \Delta f^q_{t,2}, \Delta f^q_{t,3}, u_{y,t-1}, u_{y,t}, u^1_{t-1,3}, u^1_{t,1}, u^1_{t,2}, u^1_{t,3}, \\
&u^2_{t-1,3}, u^2_{t,1}, u^2_{t,2}, u^2_{t,3}, u^3_{t-1,3}, u^3_{t,1}, u^3_{t,2}, u^3_{t,3}, u^4_{t-1,3}, u^4_{t,1}, u^4_{t,2}, u^4_{t,3})
\end{aligned}
$$

$$
M_{s_t} = \begin{pmatrix} 0_{(3,1)} \\ \varphi_{s_t,1} \\ \varphi_{s_t,2} \\ \varphi_{s_t,3} \\ 0_{(18,1)} \end{pmatrix}
\qquad
T = \begin{pmatrix}
 & I_3 & 0_{(6,2)} & & & & \\
 & W_f & & 0_{(8,4)} & 0_{(12,4)} & & \\
0_{(24,3)} & & T_0 & & & 0_{(16,4)} & \\
 & 0_{(18,3)} & & W_1 & & & 0_{(20,4)} \\
 & & 0_{(16,2)} & & W_2 & & \\
 & & & 0_{(12,4)} & & W_3 & \\
 & & & & 0_{(8,4)} & 0_{(4,4)} & W_4
\end{pmatrix}
$$

$$
R = \begin{pmatrix}
 & 0_{(3,3)} & 0_{(6,2)} & & & & \\
 & R_f & & 0_{(8,4)} & 0_{(12,4)} & & \\
0_{(24,3)} & & P_0 & & & 0_{(16,4)} & \\
 & 0_{(18,3)} & & R_1 & & & 0_{(20,4)} \\
 & & 0_{(16,2)} & & R_2 & & \\
 & & & 0_{(12,4)} & & R_3 & \\
 & & & & 0_{(8,4)} & 0_{(4,4)} & R_4
\end{pmatrix}
$$

$$\eta_t = (\eta^q_{t-1,1}, \eta^q_{t-1,2}, \eta^q_{t-1,3}, \eta^q_{t,1}, \eta^q_{t,2}, \eta^q_{t,3}, \xi_{t-1}, \xi_t, \epsilon^1_{t-1,3}, \epsilon^1_{t,1}, \epsilon^1_{t,2}, \epsilon^1_{t,3}, \epsilon^2_{t-1,3}, \epsilon^2_{t,1}, \epsilon^2_{t,2}, \epsilon^2_{t,3}, \\ \epsilon^3_{t-1,3}, \epsilon^3_{t,1}, \epsilon^3_{t,2}, \epsilon^3_{t,3}, \epsilon^4_{t-1,3}, \epsilon^4_{t,1}, \epsilon^4_{t,2}, \epsilon^4_{t,3})$$

the variance-covariance matrix of the idiosyncratic components is given by $H = 0_{(13,13)}$ and $Q_{(24\times 24)}$ is diagonal matrix equal to

$$Q = diag(0, 0, 0, 1, 1, 1, 0, \sigma^2_\xi, 0, \sigma^2_{\epsilon_1}, \sigma^2_{\epsilon_1}, \sigma^2_{\epsilon_1}, 0, \sigma^2_{\epsilon_2}, \sigma^2_{\epsilon_2}, \sigma^2_{\epsilon_2}, 0, \sigma^2_{\epsilon_3}, \sigma^2_{\epsilon_3}, \sigma^2_{\epsilon_3}, 0, \sigma^2_{\epsilon_4}, \sigma^2_{\epsilon_4}, \sigma^2_{\epsilon_4})$$

Sub-matrices inside T and R are equal to

$$W_f = \begin{pmatrix} 0 & \phi_{f2} & \phi_{f1} \\ 0 & \phi_{f1}\phi_{f2} & \phi^2_{f1} + \phi_{f2} \\ 0 & \phi^2_{f1}\phi_{f2} + \phi^2_{f2} & \phi^3_{f1} + 2\phi_{f1}\phi_{f2} \end{pmatrix} \qquad T_0 = \begin{pmatrix} 0 & 1 \\ \phi_{y2} & \phi_{y1} \end{pmatrix}$$

$$R_f = \begin{pmatrix} 1 & 0 & 0 \\ \phi_{f1} & 1 & 0 \\ \phi^2_{f1} + \phi_{f2} & \phi_{f1} & 1 \end{pmatrix} \qquad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W_i = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \phi_{i2} & \phi_{i1} \\ 0 & 0 & \phi_{i1}\phi_{i2} & \phi^2_{i1} + \phi_{i2} \\ 0 & 0 & \phi^2_{i1}\phi_{i2} + \phi^2_{i2} & \phi^3_{i1} + 2\phi_{i1}\phi_{i2} \end{pmatrix} \quad R_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \phi_{i1} & 1 & 0 \\ 0 & \phi^2_{i1} + \phi_{i2} & \phi_{i1} & 1 \end{pmatrix} \qquad i = 1, 2, 3, 4$$

while the values for $\varphi_{s_{t,1}}, \varphi_{s_{t,2}}$ and $\varphi_{s_{t,3}}$ are equal to the ones described in Appendix A.